On the Periodic Model of Uncertainty Quantification With Application to Inverse Problems

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Thanks to collaborators: Yoshihito Kazashi (Uni. Strathclyde), Frances Kuo (UNSW Sydney), Fabio Nobile (EPFL), and Ian Sloan (UNSW Sydney)

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Part I: Quasi-Monte Carlo cubature

Lattice rules

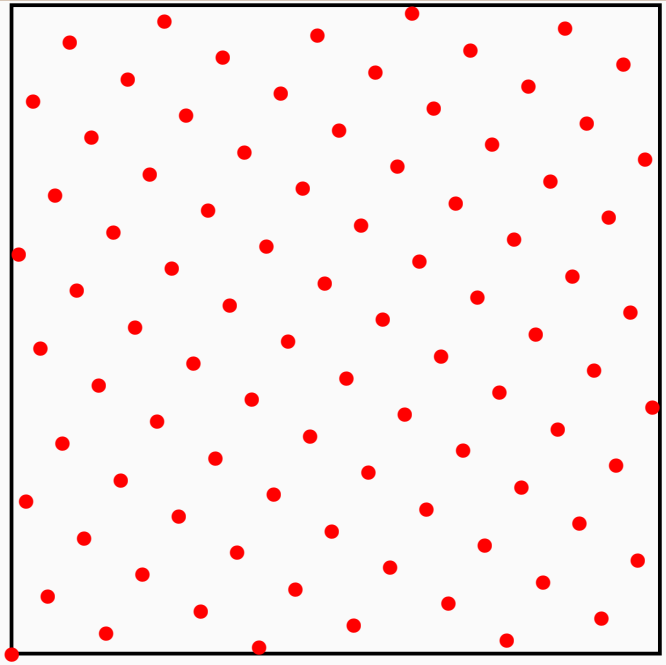
Rank-1 lattice rules

$$Q_{s,n}(f) = \frac{1}{n}\sum_{i=1}^{n} f(\boldsymbol{t}_i) \approx \int_{[0,1]^s} f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = I_s(f)$$

have the points

$$\boldsymbol{t}_i = \mathrm{mod}\left(\frac{i\boldsymbol{z}}{n}, 1\right), \quad i \in \{1, \ldots, n\},$$

where the entire point set is determined by the *generating vector* $\boldsymbol{z} \in \mathbb{N}^s$, with all components *coprime* to n.



Lattice rule with $\boldsymbol{z} = (1, 55)$ and $n = 89$ nodes in $[0, 1]^2$

Lattice rules and periodic functions are a match made in heaven!

Periodic means

$$f(y_1, y_2, \ldots, y_s) = f(y_1 + 1, y_2, \ldots, y_s) = f(y_1, y_2 + 1, \ldots, y_s) = \cdots$$

Dimension $s = 1$: the only lattice is the left-Riemann rule

For $z \in \{1, \ldots, n-1\}$, $\gcd(z, n) = 1$, it holds that

$$Q_{1,n}(f) = \frac{1}{n} \sum_{k=1}^{n} f\left(\operatorname{mod}\left(\frac{kz}{n}, 1\right)\right) = \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

Suppose $f : [0, 1) \to \mathbb{R}$ is p times continuously differentiable and periodic. Let $h = \frac{1}{n}$. Then the Euler–Maclaurin summation formula gives

$$\sum_{k=0}^{n-1} hf(kh) = \int_0^1 f(x)\,\mathrm{d}x + \sum_{k=1}^{\lfloor p/2 \rfloor} \frac{B_{2k}}{(2k)!}(f^{(2k-1)}(1) - f^{(2k-1)}(0)) - (-1)^p h^p \int_0^1 \widetilde{B}_p(x) f^{(p)}(x)\,\mathrm{d}x$$

$$\therefore \left| \int_0^1 f(x)\,\mathrm{d}x - \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) \right| = \mathcal{O}(n^{-p}).$$

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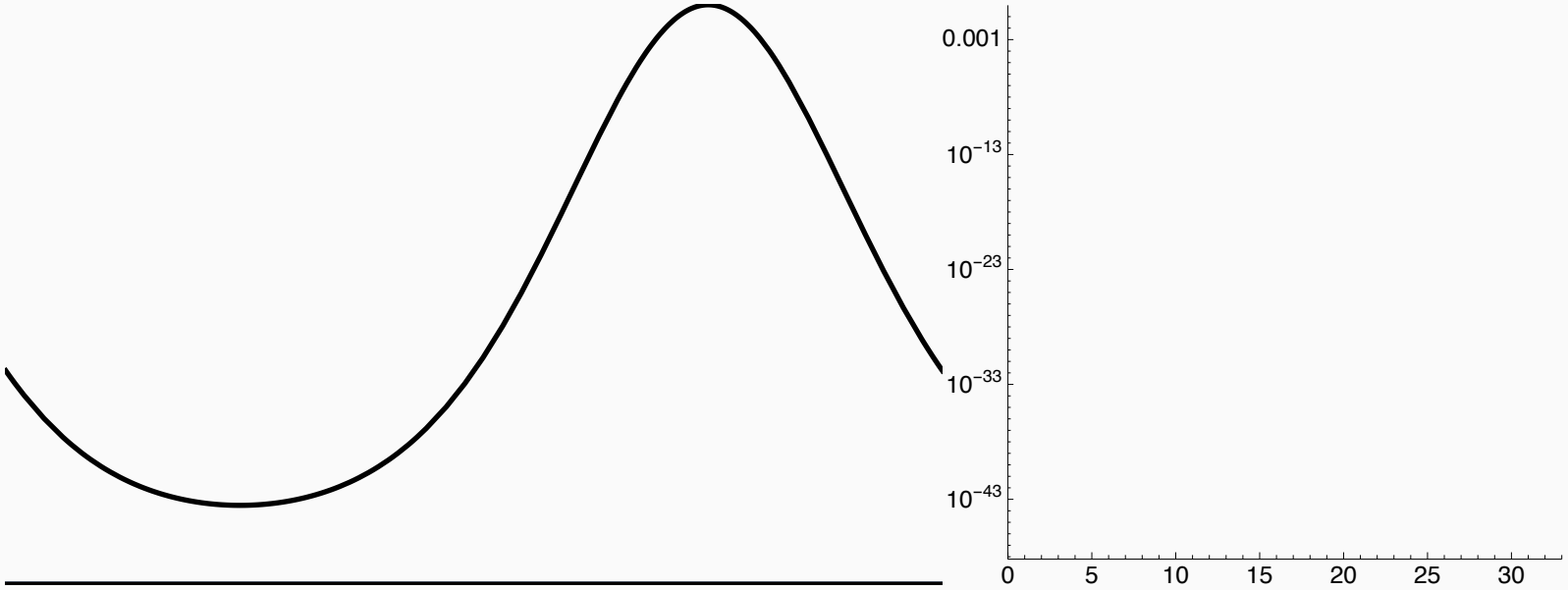
Exponential convergence for analytic, periodic functions

$$\int_0^1 \exp(-\sin(2\pi x))\,\mathrm{d}x$$



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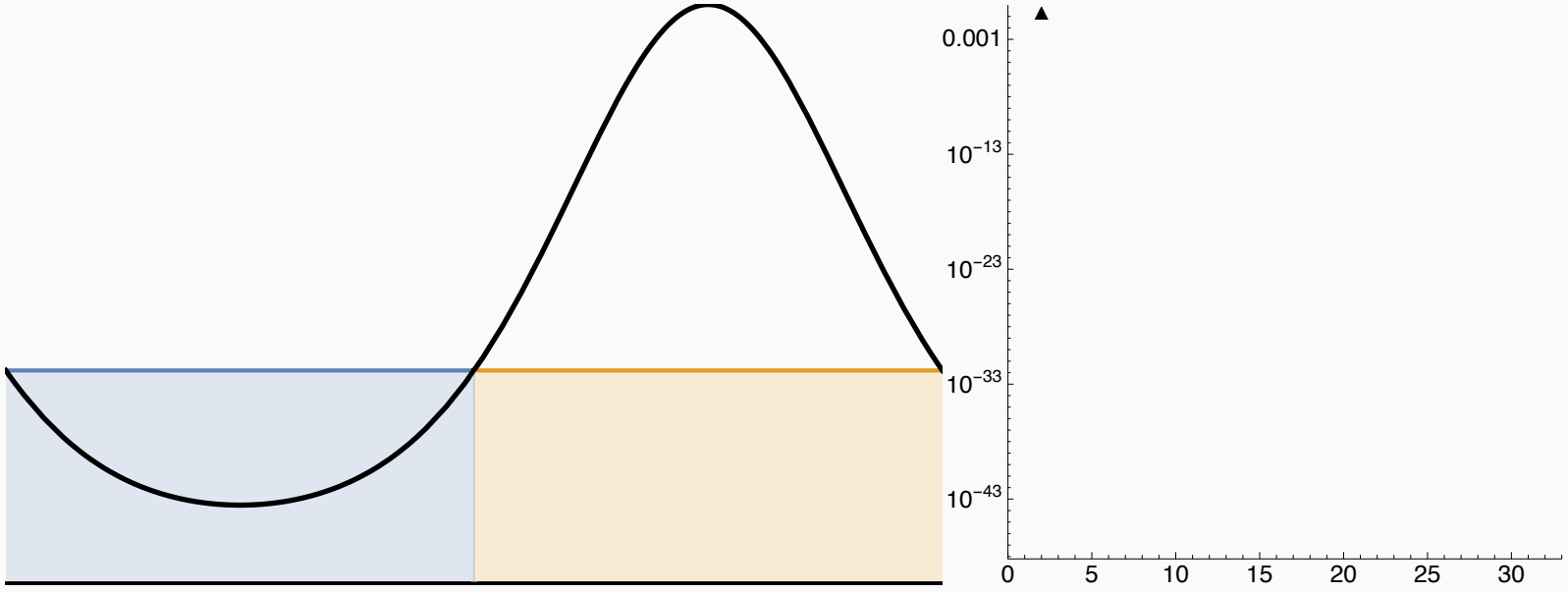

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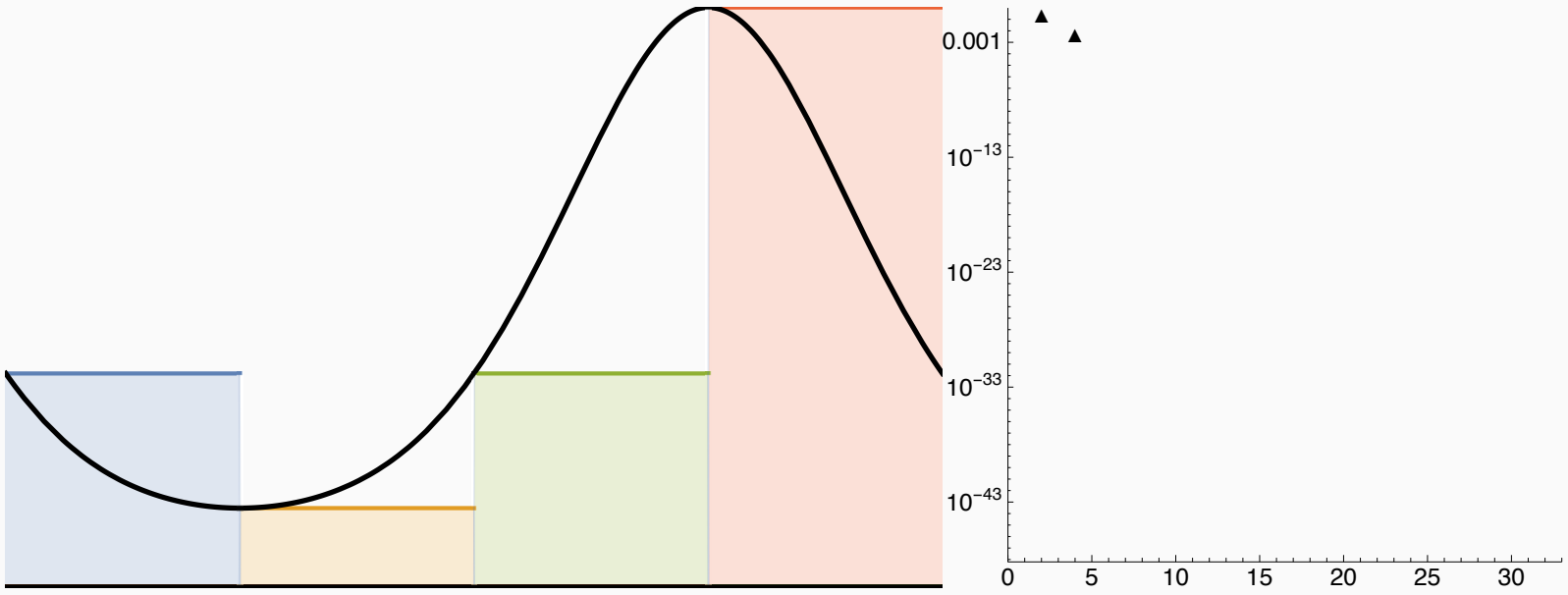

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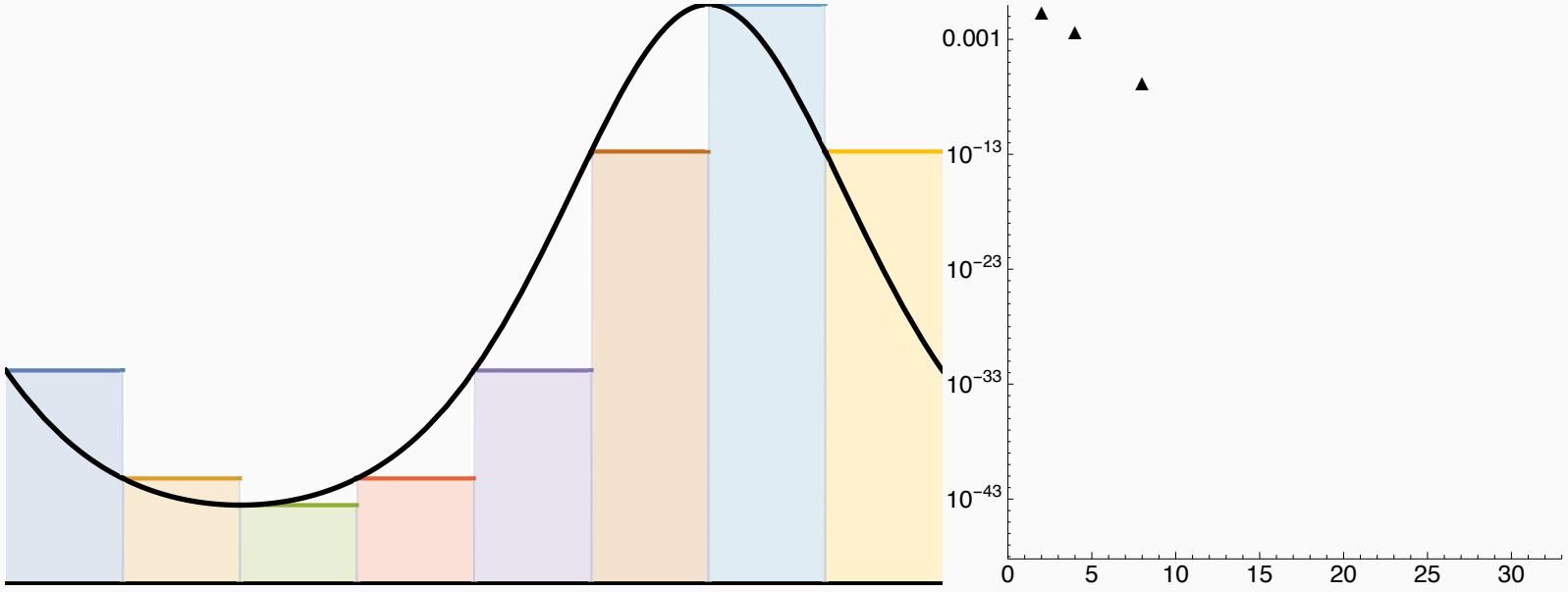

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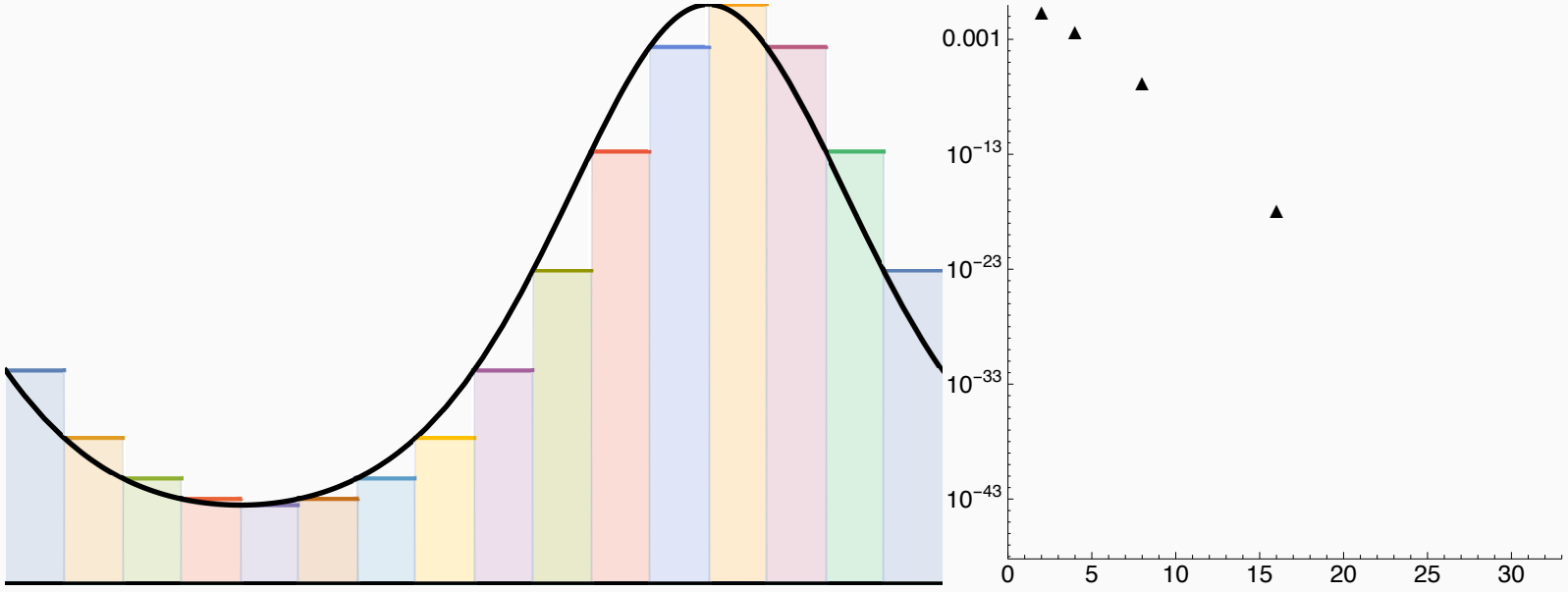

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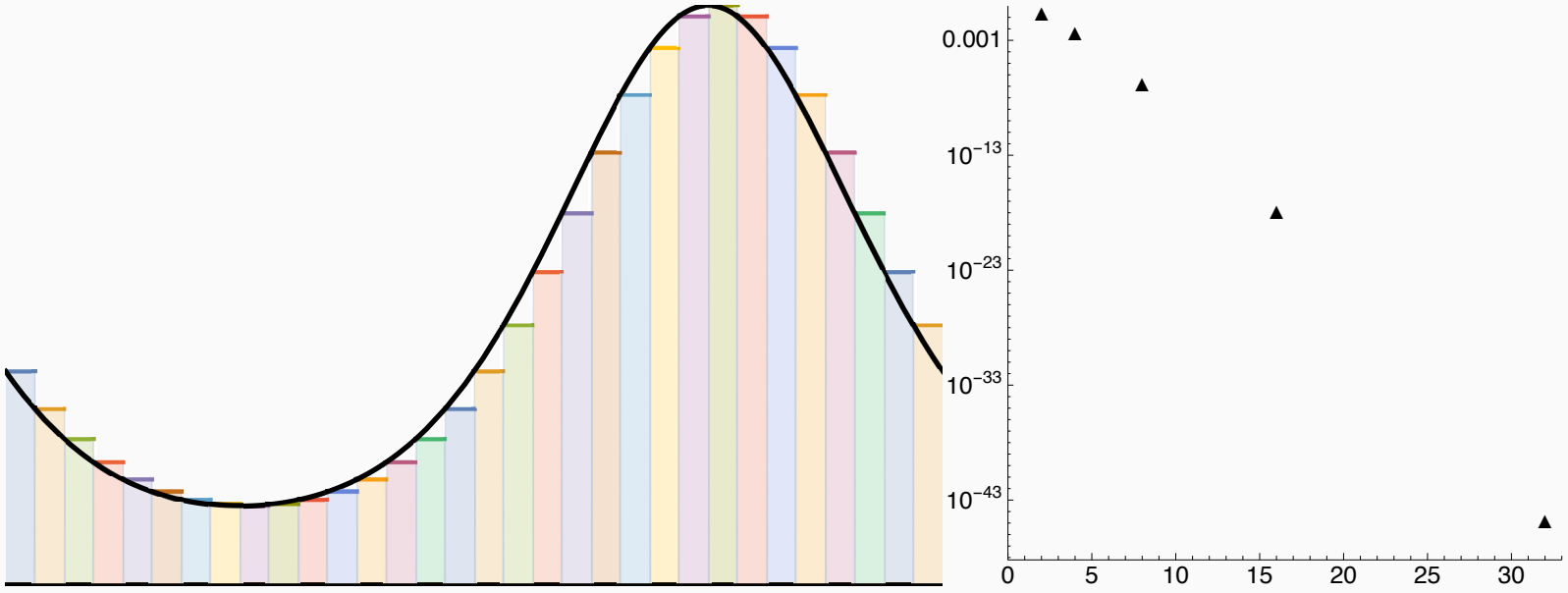

Can we observe exponential convergence with lattice rules for analytic, periodic functions when dimension $s = 2$?

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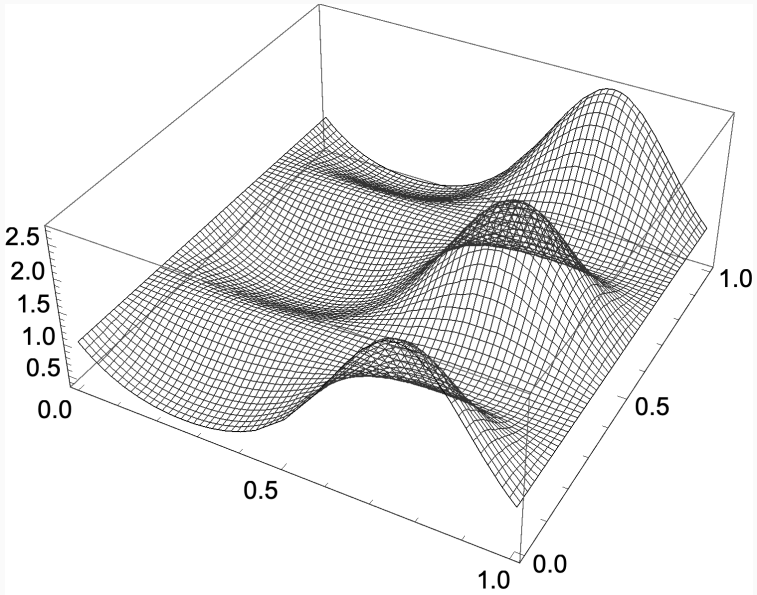

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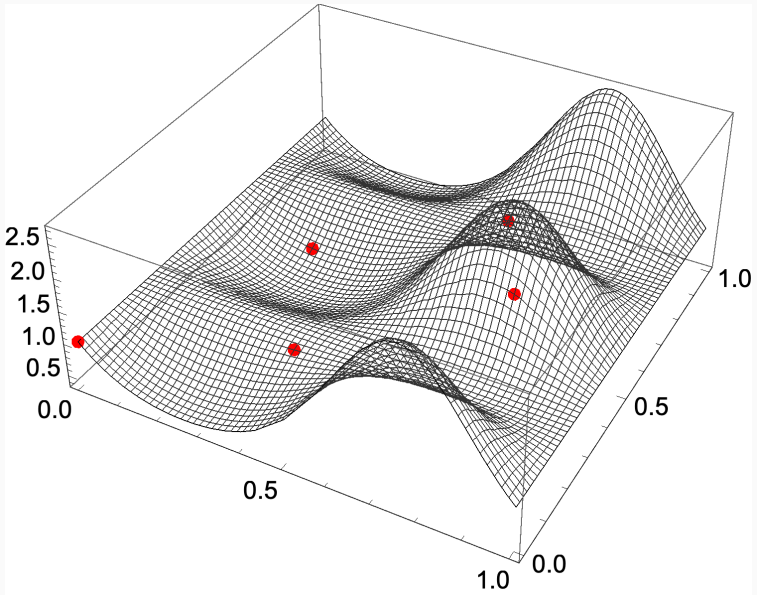

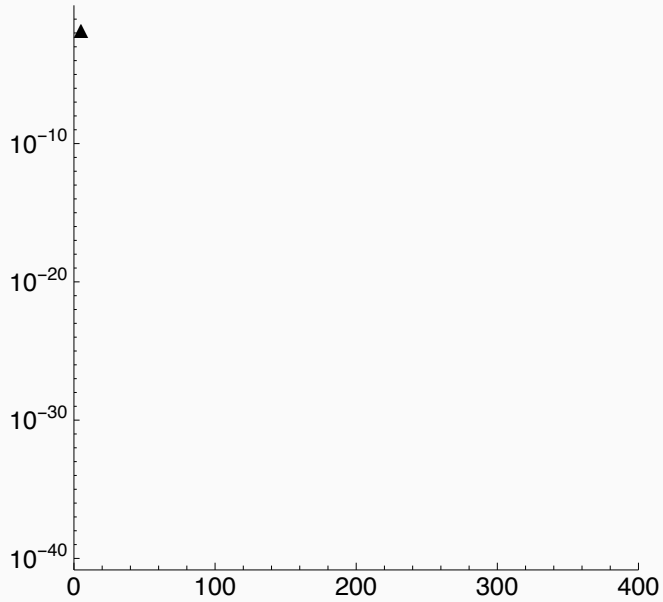

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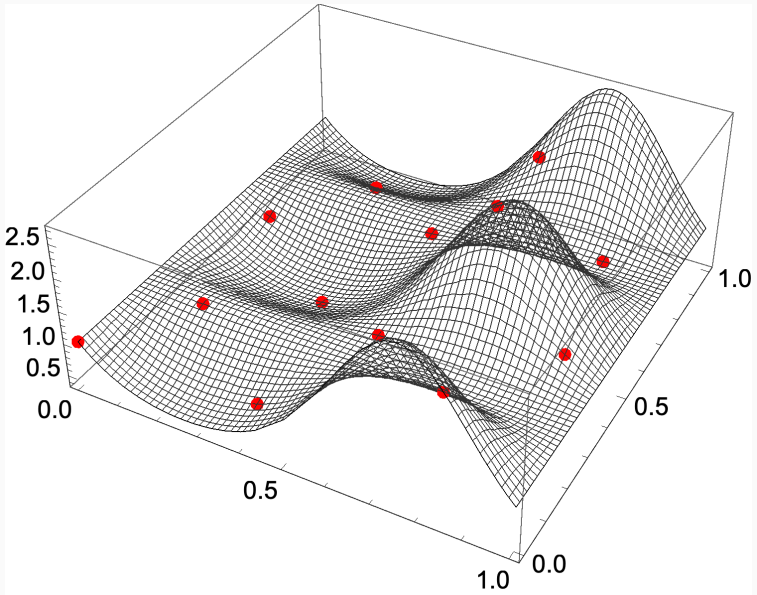

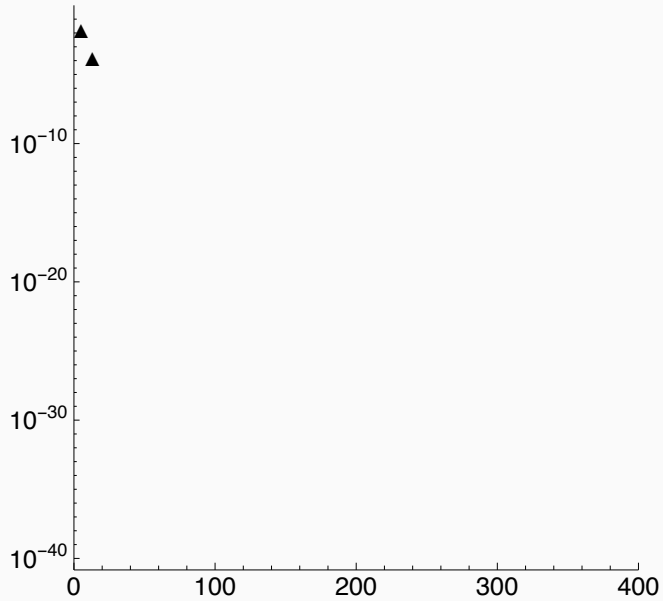

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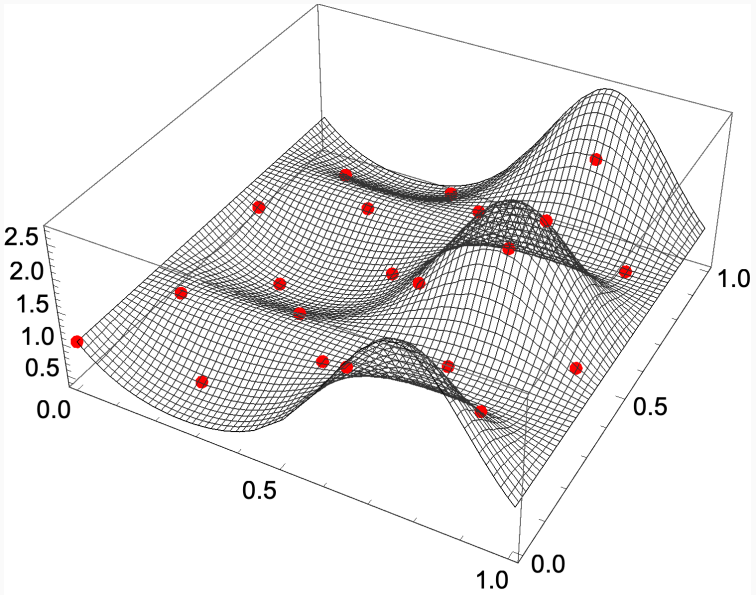

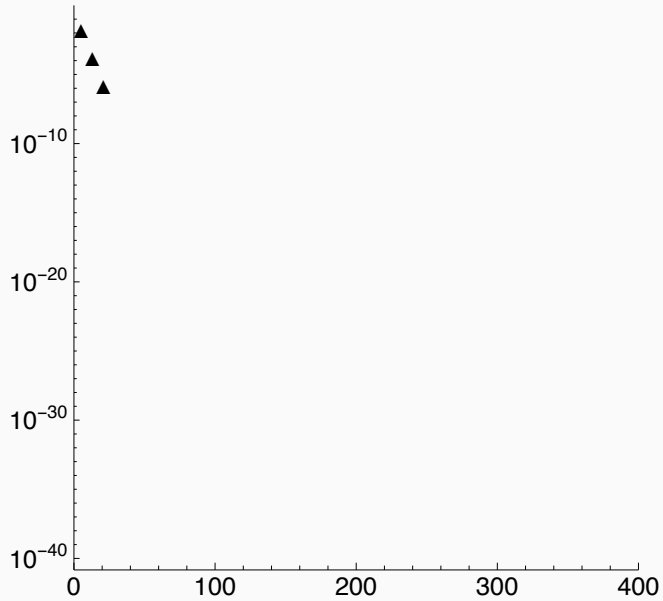

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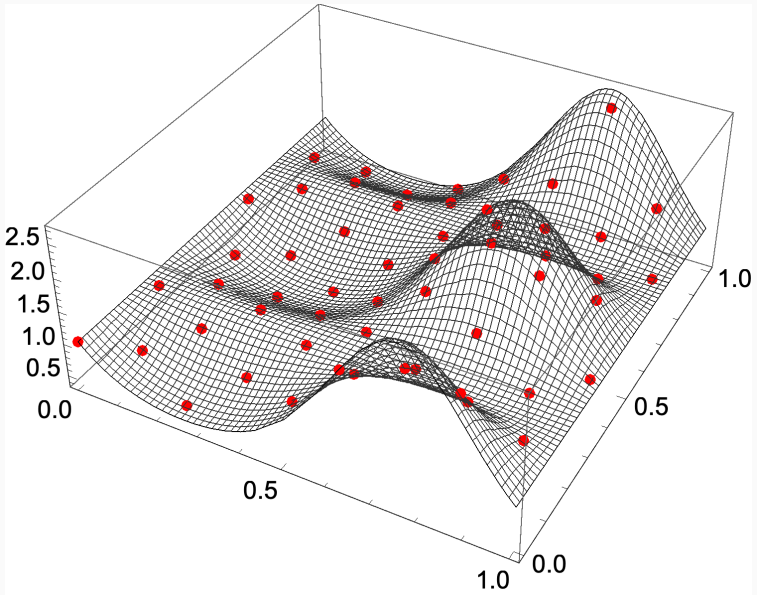

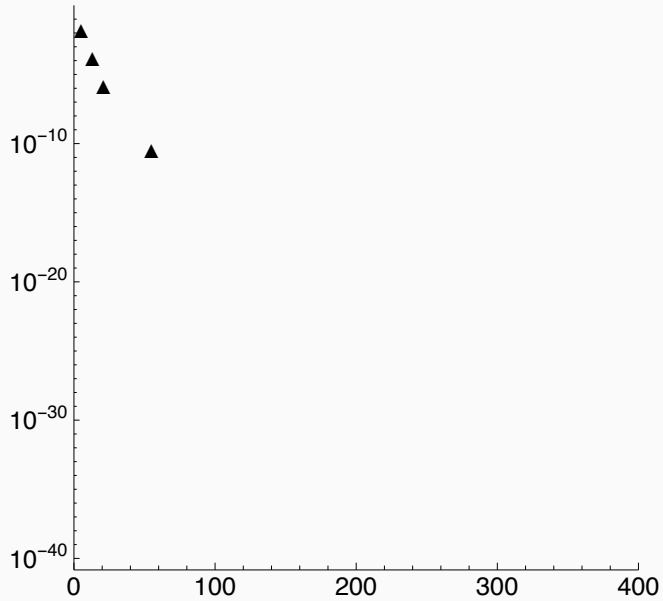

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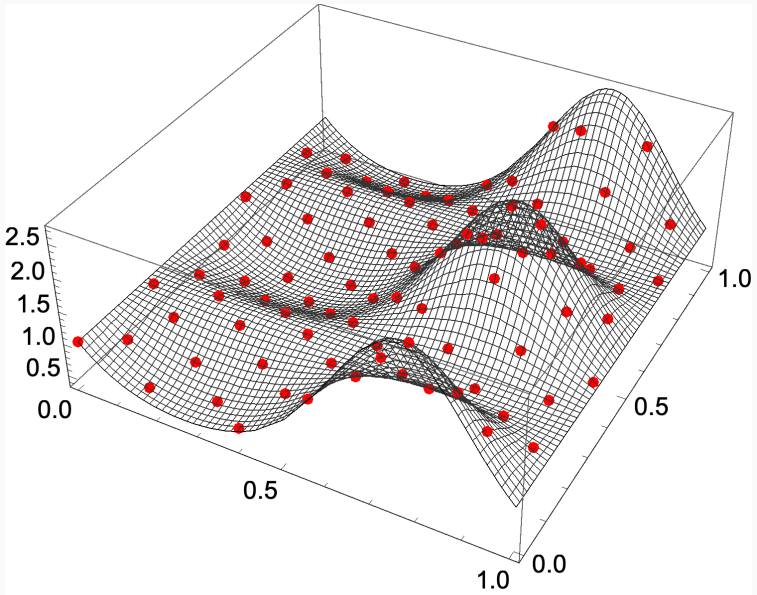

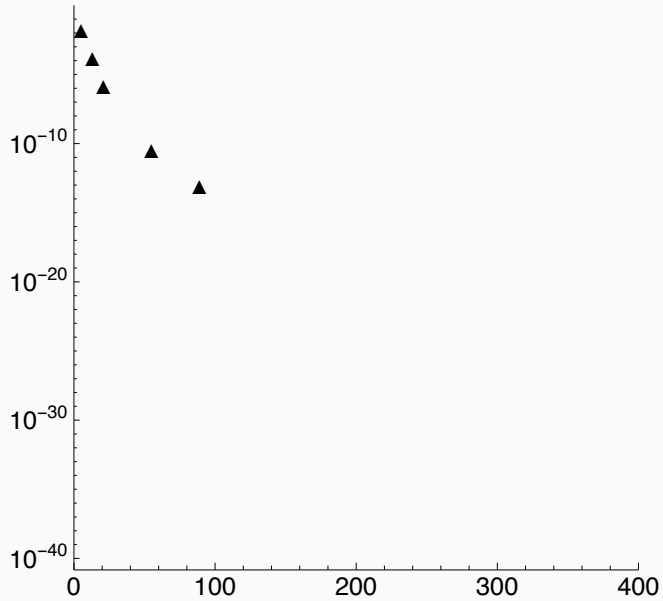

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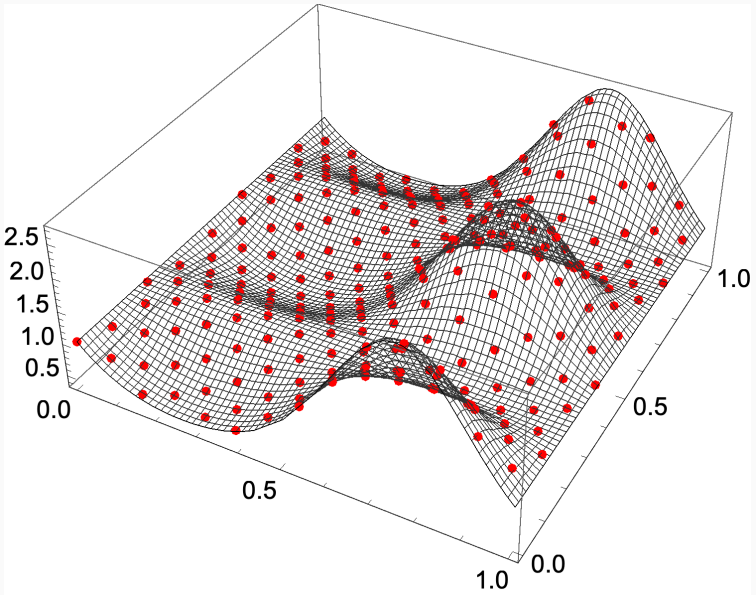

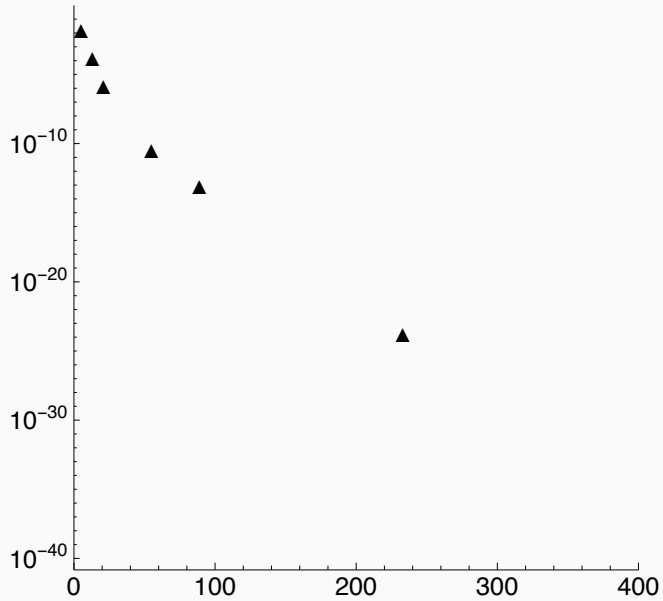

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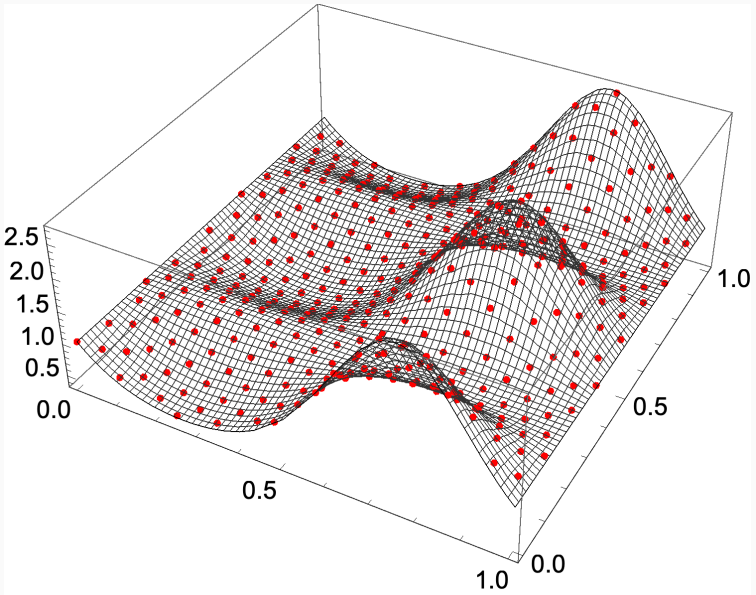
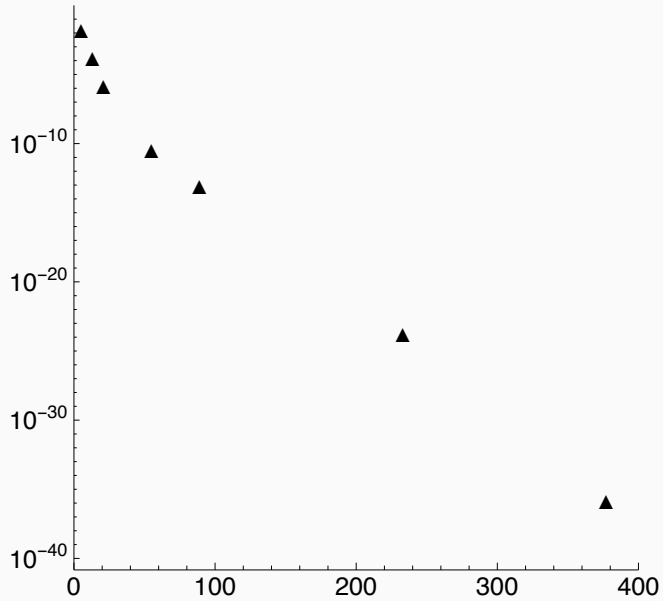
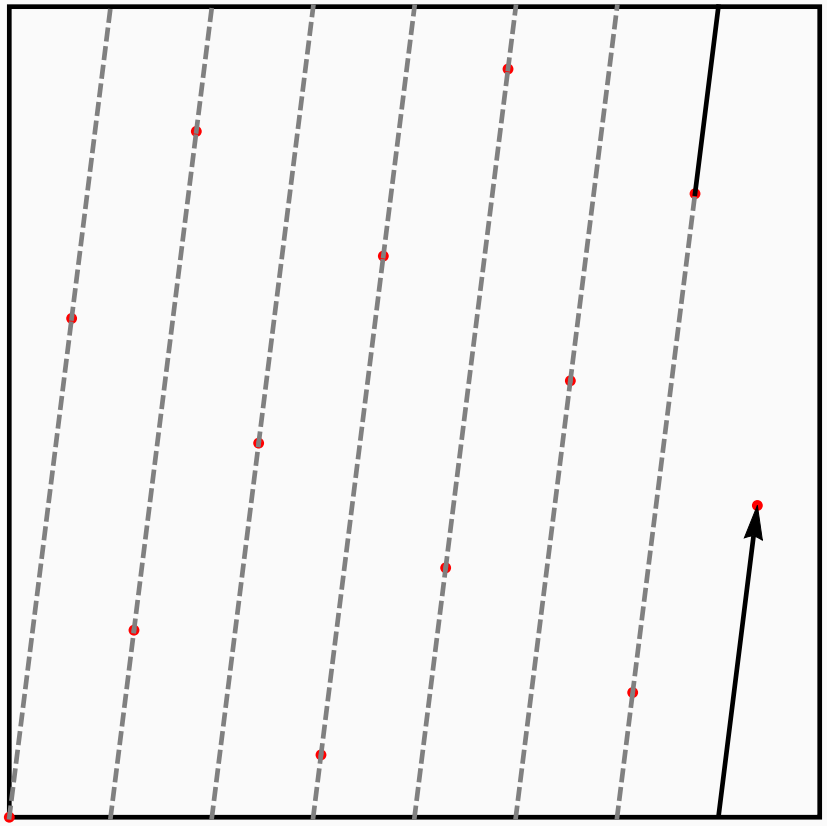






A suitable generating vector for an integrand satisfying certain smoothness properties can be found using a *component-by-component* (CBC) algorithm (Nuyens and Cools 2006; Kuo, Nuyens, and Cools 2006).

- For integrands belonging to certain weighted Sobolev spaces of smooth functions, the CBC algorithm can be used to produce a generating vector satisfying a rigorous error bound. As input, the CBC algorithm takes the weights and smoothness parameter of the Sobolev space (and number of QMC nodes n).
- Fast CBC: FFT can be used to reduce the computational complexity of the CBC algorithm.

Part II: The periodic model of uncertainty quantification for PDEs

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $D \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, a bounded physical domain with Lipschitz boundary.

Elliptic PDE with uncertain/random coefficient

Find $u\colon D \times \Omega \to \mathbb{R}$ that satisfies

$$\begin{aligned} -\nabla \cdot (a(\boldsymbol{x}, \boldsymbol{\omega}) \nabla u(\boldsymbol{x}, \boldsymbol{\omega})) &= f(\boldsymbol{x}) && \text{for } \boldsymbol{x} \in D, \\ + \text{boundary conditions} && && \text{on } \partial D \end{aligned}$$

for almost all events $\boldsymbol{\omega} \in \Omega$. Here, the diffusion coefficient $a(\cdot, \boldsymbol{\omega}) \in L^\infty_+(D)$ is *uncertain*.

In forward uncertainty quantification, one is interested in computing certain response statistics of the solution, usually $\mathbb{E}[u]$ *or* $\mathbb{E}[G(u)]$ *and* $\mathrm{Var}[u]$ *or* $\mathrm{Var}[G(u)]$*, where* G *is a (linear) functional representing some quantity of interest derived from the solution.*

Depending on the application, two common models for the random field A that appear in the literature are

- uniform and affine;
- lognormal.

Background

A popular model in the literature: the uniform and affine model

For $\boldsymbol{x} \in D$ and $\boldsymbol{\omega} \in \Omega$,

$$a(\boldsymbol{x}, \boldsymbol{\omega}) = \bar{a}(\boldsymbol{x}) + \sum_{j \geq 1} Y_j(\omega)\psi_j(\boldsymbol{x}), \quad Y_j \text{ i.i.d. uniform on } [-\tfrac{1}{2}, \tfrac{1}{2}].$$

Computing $\mathbb{E}[u(\boldsymbol{x}, \cdot)]$ (or some quantity of interest $\mathbb{E}[G(u)]$) using

- Rank-1 lattice cubature rules with random shifts
  $\Rightarrow$ cubature error $\mathcal{O}(n^{-1+\varepsilon})$ at best. (Kuo, Schwab, Sloan 2012)
- Interlaced polynomial lattice rules
  $\Rightarrow$ higher order convergence $\mathcal{O}(n^{-1/p})$ for some $0 < p < 1$ (p is a summability exponent s.t. $(\|\psi_j\|_{L^\infty})_{j \geq 1} \in \ell^p$). (Dick, Kuo, Le Gia, Nuyens, Schwab 2014)

Periodic model of UQ

In this talk, we instead model the uncertainty in the diffusion coefficient as follows.

For $\boldsymbol{x} \in D$ and $\boldsymbol{\omega} \in \Omega$,

$$a(\boldsymbol{x}, \boldsymbol{\omega}) = \bar{a}(\boldsymbol{x}) + \sum_{j \geq 1} \Theta(Y_j(\omega))\psi_j(\boldsymbol{x}), \quad Y_j \text{ i.i.d. uniform on } [-\tfrac{1}{2}, \tfrac{1}{2}]$$

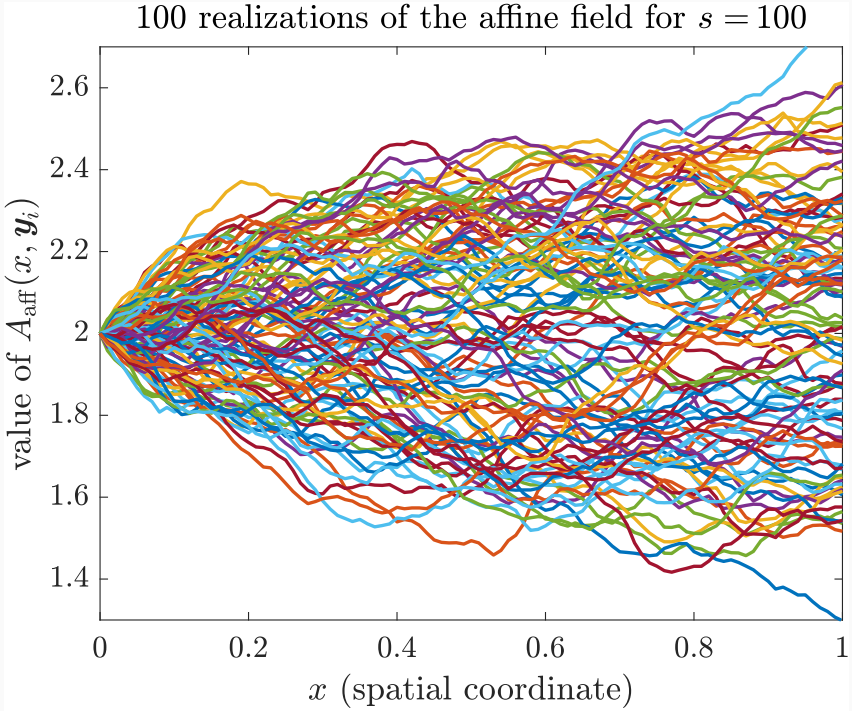
with the special choice $\Theta(y) = \sin(2\pi y)$.

- Note that $Z(\omega) := \sin(2\pi Y(\omega))$ has the probability density $\frac{1}{\pi}\frac{1}{\sqrt{1-z^2}}$ on $[-1, 1]$, i.e, $Z \sim \mathrm{Arcsine}(-1, 1)$.
- We can match the mean and covariance of a with the "uniform model" by choosing $\Theta(y) = \frac{1}{\sqrt{6}}\sin(2\pi y)$.
- Note that the periodicity is only assumed for the *random/uncertain* variable!

Affine vs. periodic

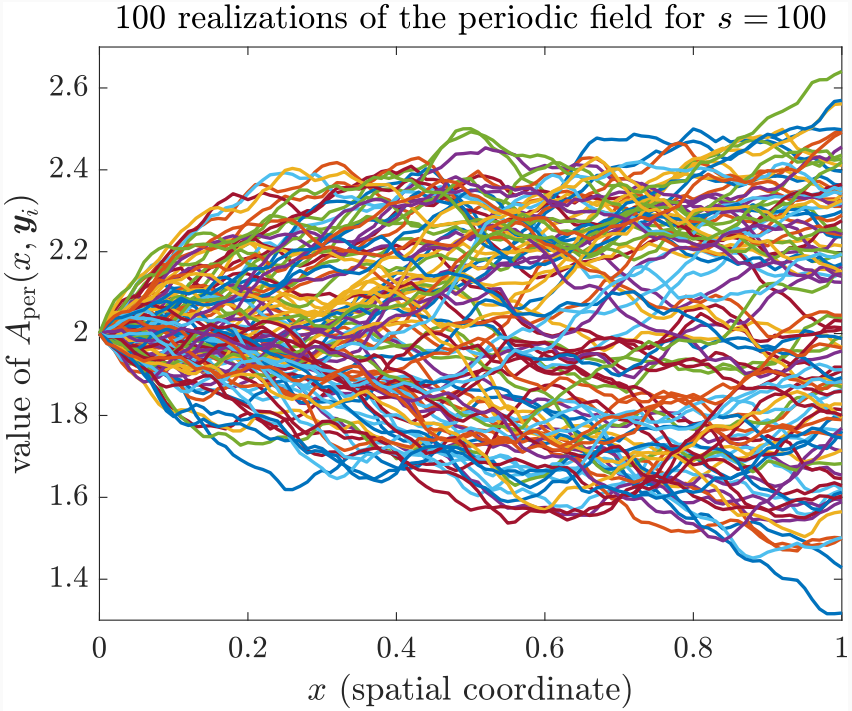
Affine

$$a(x, \boldsymbol{y}) = \bar{a}(x) + \sum_{j=1}^{100} y_j \psi_j(x)$$



Periodic

$$a(x, \boldsymbol{y}) = \bar{a}(x) + \frac{1}{\sqrt{6}} \sum_{j=1}^{100} \sin(2\pi y_j)\psi_j(x)$$



$$\bar{a}(x) = 2, \quad \psi_j(x) = j^{-3/2} \sin((j - \tfrac{1}{2})\pi x), \quad x \in [0, 1]$$

Let $U := [-1/2, 1/2]^{\mathbb{N}}$ and $D \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, a nonempty bounded Lipschitz domain. For the parametric PDE

$$\begin{cases} -\nabla \cdot (a(\boldsymbol{x}, \boldsymbol{y}) \nabla u(\boldsymbol{x}, \boldsymbol{y})) = f(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in D,\ \boldsymbol{y} \in U \\ u(\boldsymbol{x}, \boldsymbol{y}) = 0 & \text{for } \boldsymbol{x} \in \partial D,\ \boldsymbol{y} \in U, \end{cases}$$

with $u(\cdot, \boldsymbol{y}) \in H_0^1(D)$, $f \in H^{-1}(D)$, and

$$a(\boldsymbol{x}, \boldsymbol{y}) = \bar{a}(\boldsymbol{x}) + \sum_{j=1}^{s} \sin(2\pi y_j) \psi_j(\boldsymbol{x}),$$

with assumptions

- $0 < a_{\min} \le a(\boldsymbol{x}, \boldsymbol{y}) \le a_{\max} < \infty$ for all $\boldsymbol{x} \in D$, $\boldsymbol{y} \in U$
- $\sum_{j=1}^{\infty} \|\psi_j\|_{L^\infty}^p < \infty$ for some $p \in (0, 1)$
- $\|\psi_1\|_{L^\infty} \ge \|\psi_2\|_{L^\infty} \ge \cdots$

[K–Kuo–Sloan 2020] showed that there exists a constructible lattice rule satisfying the QMC cubature error

$$|I_s(G(u)) - Q_{n,s}(G(u))| \le C n^{-1/p} \quad \text{with constant } C > 0 \text{ independent of } s,$$

for any linear quantity of interest $G : H_0^1(D) \to \mathbb{R}$.

Numerical example: QMC for PDE [K–Kuo–Sloan (2020)]

Let us consider the PDE problem

$$-\nabla \cdot (a_{\mathrm{per}}(\boldsymbol{x}, \boldsymbol{y}) \nabla u(\boldsymbol{x}, \boldsymbol{y})) = x_2, \quad u(\cdot, \boldsymbol{y})|_{\partial D} = 0,$$

in the physical domain $D = (0,1)^2$ with the diffusion coefficient

$$a_{\mathrm{per}}(\boldsymbol{x}, \boldsymbol{y}) = 2 + \sum_{j=1}^{100} \sin(2\pi y_j)\psi_j(x), \quad y_j \in [-\tfrac{1}{2}, \tfrac{1}{2}],$$

where $\psi_j(\boldsymbol{x}) = \frac{c}{\sqrt{6}} j^{-\theta} \sin(j\pi x_1) \sin(j\pi x_2)$. **Note that** $\|\psi_j\|_{L^\infty} \propto j^{-\theta}$.

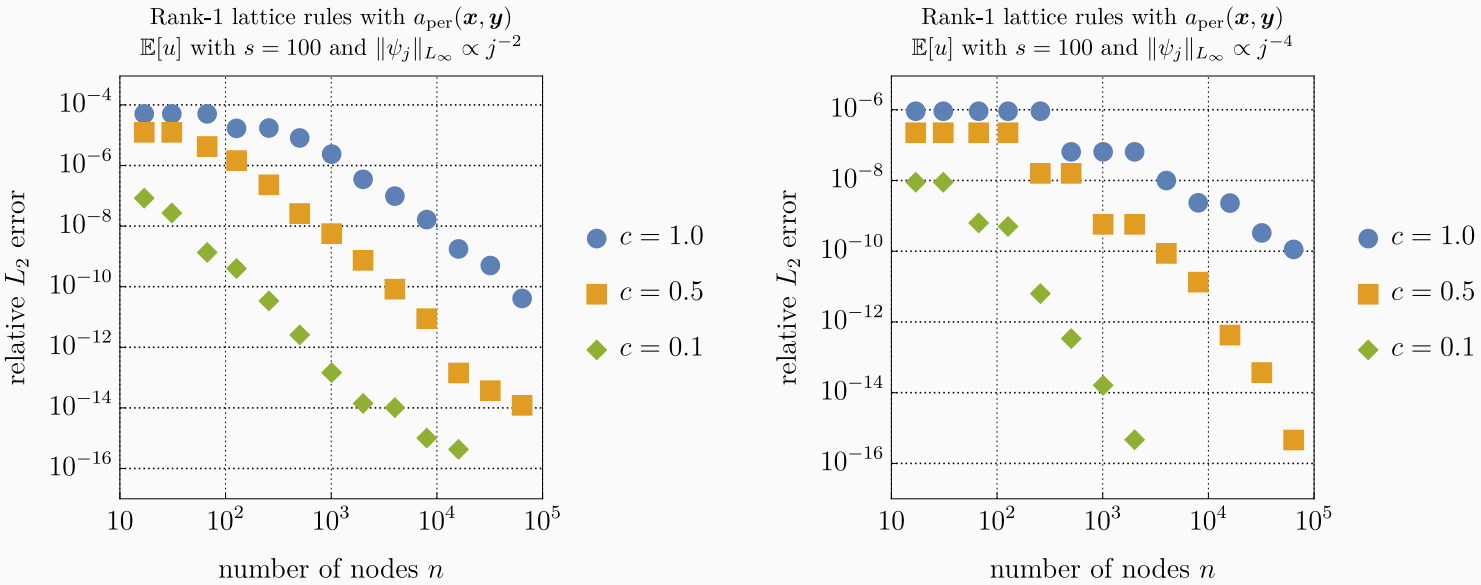


Figure 7: Left: $\theta = 2$. Right: $\theta = 4$.

Part III: Kernel interpolation over lattice point sets

Let us continue the study of our elliptic model PDE problem.

In [K–Kazashi–Kuo–Nobile–Sloan (2022)], we studied *kernel interpolation of smooth, periodic functions based on lattice point sets*. We considered the following setting:

- Let $\alpha \geq 1$ be an integer and let $H := H_{s,\alpha,\boldsymbol{\gamma}}$ be the Hilbert space containing absolutely continuous, somewhat smooth periodic functions $f : [0,1)^s \to \mathbb{R}$ endowed with the norm

$$\|f\|_H^2 := \sum_{\mathfrak{u} \subseteq \{1:s\}} \frac{1}{(2\pi)^{2\alpha|\mathfrak{u}|}\gamma_{\mathfrak{u}}} \int_{[0,1]^{|\mathfrak{u}|}} \left| \int_{[0,1]^{s-|\mathfrak{u}|}} \left(\prod_{j \in \mathfrak{u}} \frac{\partial^\alpha}{\partial y_j^\alpha} \right) f(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}_{-\mathfrak{u}} \right|^2 \mathrm{d}\boldsymbol{y}_{\mathfrak{u}},$$

  provided that f has mixed partial derivatives of order α.

The space H is actually a *reproducing kernel Hilbert space* (RKHS), with an explicitly known and analytically simple reproducing kernel:

$$K(\boldsymbol{y}, \boldsymbol{y}') := \sum_{u \subseteq \{1:s\}} \gamma_u \prod_{j \in u} \eta_\alpha(y_j, y_j'),$$

where

$$\eta_\alpha(y, y') = \frac{(2\pi)^{2\alpha}}{(-1)^{\alpha+1}(2\alpha)!} B_{2\alpha}(\mathrm{frac}(y - y')), \quad y, y' \in [0, 1],$$

where $B_2(y) = y^2 - y + \frac{1}{6}$, $B_4(y) = y^4 - 2y^3 + y^2 - \frac{1}{30}$, and so on, are the *Bernoulli polynomials*. In particular,

$$\langle f, K(\cdot, \boldsymbol{y}) \rangle_H = f(\boldsymbol{y}) \quad \text{for all } f \in H \text{ and } \boldsymbol{y} \in [0, 1]^s.$$

Example: If $(\gamma_u)_{u \subseteq \{1,\ldots,s\}}$ are *product weights*, i.e.,

$$\gamma_u := \prod_{j \in u} \gamma_j, \quad u \subseteq \{1, \ldots, s\},$$

then

$$K(\boldsymbol{y}, \boldsymbol{y}') = \prod_{j=1}^{s} (1 + \gamma_j \eta_\alpha(y_j, y_j')).$$

Suppose that one is interested in finding an approximation for the function $f \in H$ based on the point evaluations $f(\boldsymbol{t}_1), \ldots, f(\boldsymbol{t}_n)$, $\boldsymbol{t}_j \in [0,1]^s$. We introduce the *kernel interpolant*

$$f_n(\boldsymbol{y}) := \sum_{k=1}^{n} c_k K(\boldsymbol{t}_k, \boldsymbol{y}), \quad \boldsymbol{t}_k := \operatorname{mod}\left(\frac{k\boldsymbol{z}}{n}, 1\right), \tag{1}$$

and require the interpolation property $f_n(\boldsymbol{t}_k) = f(\boldsymbol{t}_k)$ for hold for all $k = 1, \ldots, n$. Then the coefficients can be solved from the linear system

$$\boldsymbol{K}\boldsymbol{c} = \boldsymbol{f},$$

where $\boldsymbol{c} := [c_1, \ldots, c_n]^{\mathrm{T}}$ are the coefficients in (1) and

$$K_{k,\ell} = K(\boldsymbol{t}_k, \boldsymbol{t}_\ell) \quad \text{and} \quad \boldsymbol{f} := [f(\boldsymbol{t}_1), \ldots, f(\boldsymbol{t}_n)]^{\mathrm{T}}.$$

Note that $K_{k,\ell} = K(\frac{(k-\ell)\boldsymbol{z}}{n}, \boldsymbol{0})$, i.e., $\boldsymbol{K}$ is a *circulant matrix* $\Rightarrow$

$$\boldsymbol{c} = \texttt{ifft}\big(\texttt{fft}(\boldsymbol{f})./\texttt{fft}(\boldsymbol{K}_{:,1})\big)$$

This can be computed in $\mathcal{O}(n \log n)$ time!

The kernel interpolant is cheap to construct!

In analogy to the cubature setting, the PDE problem

$$\begin{cases} -\nabla \cdot (a(\boldsymbol{x}, \boldsymbol{y}) \nabla u(\boldsymbol{x}, \boldsymbol{y})) = f(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in D,\ \boldsymbol{y} \in U, \\ u(\boldsymbol{x}, \boldsymbol{y}) = 0 & \text{for } \boldsymbol{x} \in \partial D,\ \boldsymbol{y} \in U, \end{cases}$$

with $u(\cdot, \boldsymbol{y}) \in H_0^1(D)$, $f \in H^{-1}(D)$, and

$$a(\boldsymbol{x}, \boldsymbol{y}) = \bar{a}(\boldsymbol{x}) + \sum_{j=1}^{s} \sin(2\pi y_j) \psi_j(\boldsymbol{x})$$

and assumptions

- $0 < a_{\min} \le a(\boldsymbol{x}, \boldsymbol{y}) \le a_{\max} < \infty$ for all $\boldsymbol{x} \in D$, $\boldsymbol{y} \in U$
- $\sum_{j=1}^{\infty} \|\psi_j\|_{L^\infty}^p < \infty$ for some $p \in (0, 1)$
- $\|\psi_1\|_{L^\infty} \ge \|\psi_2\|_{L^\infty} \ge \cdots$

[K–Kazashi–Kuo–Nobile–Sloan 2022] showed that there exists a sequence of SPOD weights (entering both the expression of the kernel K in the interpolant and as inputs to a CBC algorithm) and a constructible lattice rule satisfying the kernel approximation error

$$\|u - u_n\|_{L^2(U \times D)} = \mathcal{O}(n^{-\frac{1}{2p} + \frac{1}{4}}) \quad \text{with constant } C > 0 \text{ independent of } s.$$

Kernel approximation for PDE: L^2 error

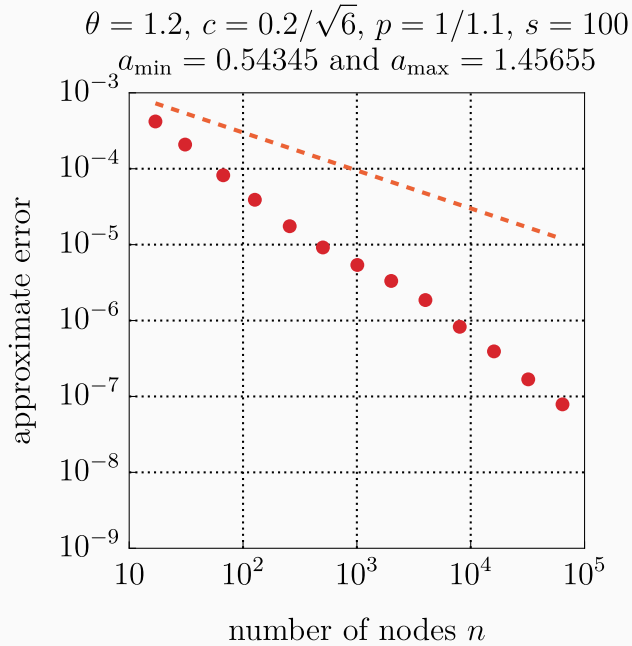
Let us consider the PDE problem

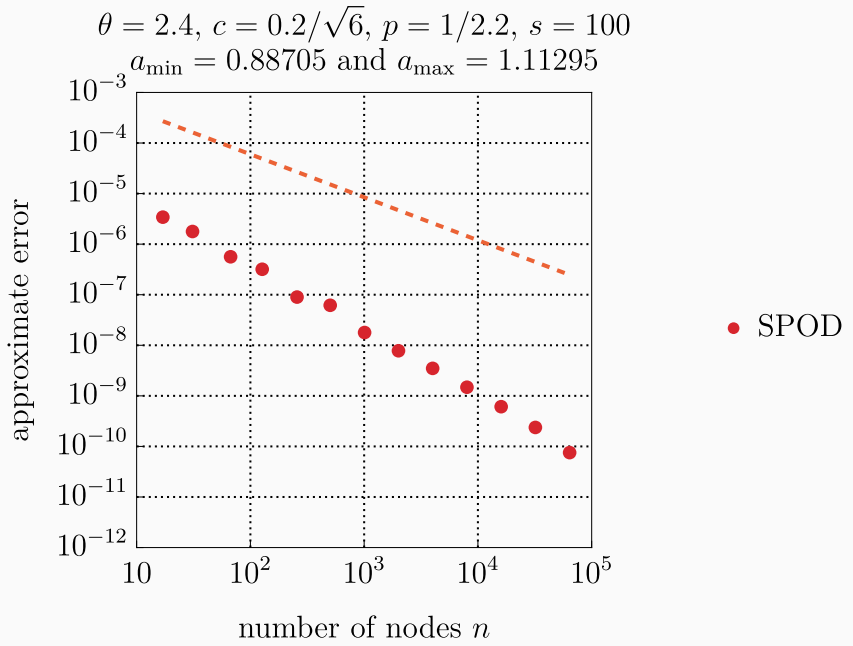
$$-\nabla \cdot (a_{\text{per}}(\boldsymbol{x}, \boldsymbol{y}) \nabla u(\boldsymbol{x}, \boldsymbol{y})) = x_2, \quad u(\cdot, \boldsymbol{y})|_{\partial D} = 0,$$

in the physical domain $D = (0,1)^2$ with the diffusion coefficient

$$a_{\text{per}}(\boldsymbol{x}, \boldsymbol{y}) = 1 + \sum_{j=1}^{100} \sin(2\pi y_j)\psi_j(x), \quad y_j \in [0,1],$$

where $\psi_j(\boldsymbol{x}) = cj^{-\theta} \sin(j\pi x_1) \sin(j\pi x_2)$. **Note that** $\|\psi_j\|_{L^\infty} \propto j^{-\theta}$.





Reducing the computational complexity

The SPOD weights used in the construction of the kernel interpolant were

$$\gamma_{\mathfrak{u}} := \sum_{\boldsymbol{m}_{\mathfrak{u}} \in \{1:\alpha\}^{|\mathfrak{u}|}} (|\boldsymbol{m}_{\mathfrak{u}}|!)^{\frac{2}{1+\lambda}} \prod_{j \in \mathfrak{u}} \left(\frac{b_j^{m_j} S(\alpha, m_j)}{\sqrt{2\mathrm{e}^{1/\mathrm{e}} \zeta(2\alpha\lambda)}} \right)^{\frac{2}{1+\lambda}}. \tag{2}$$

- the cost to obtain the generating vector $\boldsymbol{z}$ is $\mathcal{O}(s\, n \log n + s^3\, \alpha^2\, n)$;
- the cost of evaluating the kernel interpolant is $\mathcal{O}(s^2\, \alpha^2\, n)$.

New idea (see Ian's talk on Friday): leave out the order-dependent part $(|\boldsymbol{m}_{\mathfrak{u}}|!)^{\frac{2}{1+\lambda}}$ in (2), get

$$\tilde{\gamma}_{\mathfrak{u}} := \sum_{\boldsymbol{m}_{\mathfrak{u}} \in \{1:\alpha\}^{|\mathfrak{u}|}} \prod_{j \in \mathfrak{u}} \left(\frac{b_j^{m_j} S(\alpha, m_j)}{\sqrt{2\mathrm{e}^{1/\mathrm{e}} \zeta(2\alpha\lambda)}} \right)^{\frac{2}{1+\lambda}} = \prod_{j \in \mathfrak{u}} \left(\sum_{m=1}^{\alpha} \left(\frac{b_j^{m} S(\alpha, m)}{\sqrt{2\mathrm{e}^{1/\mathrm{e}} \zeta(2\alpha\lambda)}} \right)^{\frac{2}{1+\lambda}} \right).$$

These are *product weights* (**"serendipitous weights"**), where

- the cost to obtain the generating vector $\boldsymbol{z}$ is $\mathcal{O}(s\, n \log n)$;
- the cost of evaluating the kernel interpolant is $\mathcal{O}(s\, n)$.

Kernel approximation for PDE: L^2 error (redux)

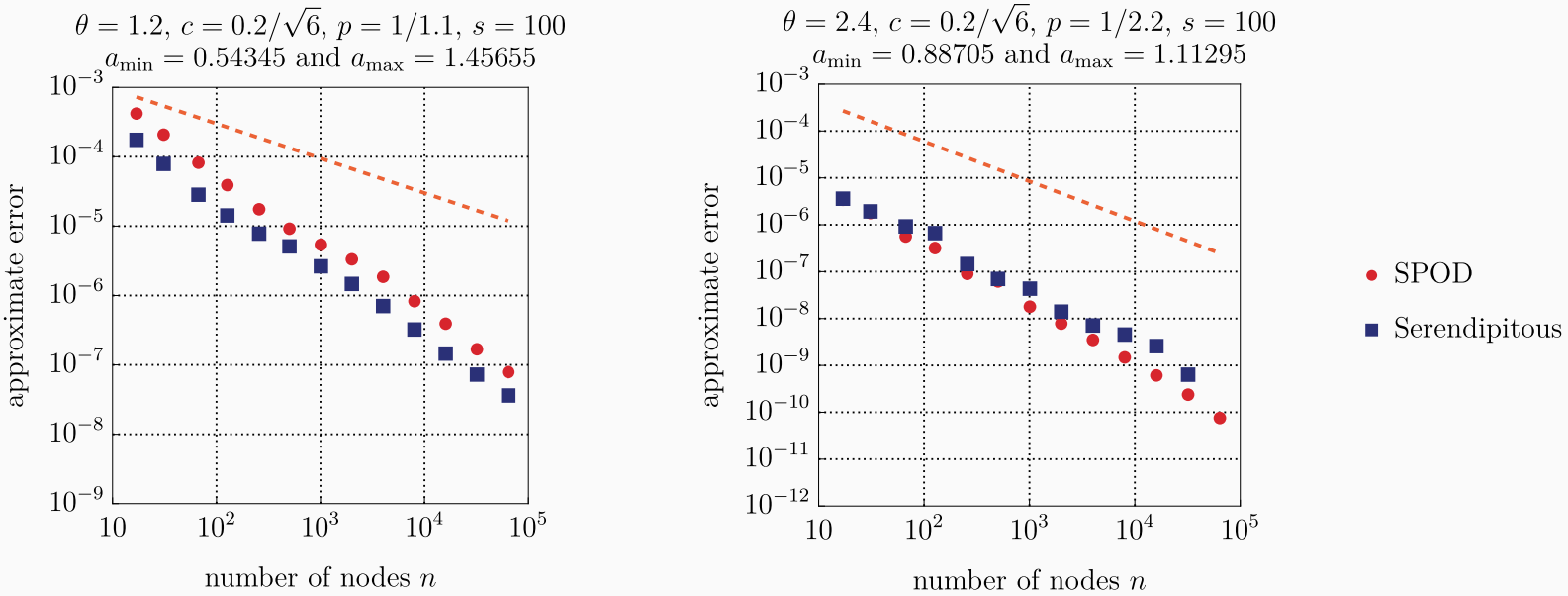
Let us consider the PDE problem

$$-\nabla \cdot (a_{\text{per}}(\boldsymbol{x}, \boldsymbol{y}) \nabla u(\boldsymbol{x}, \boldsymbol{y})) = x_2, \quad u(\cdot, \boldsymbol{y})|_{\partial D} = 0,$$

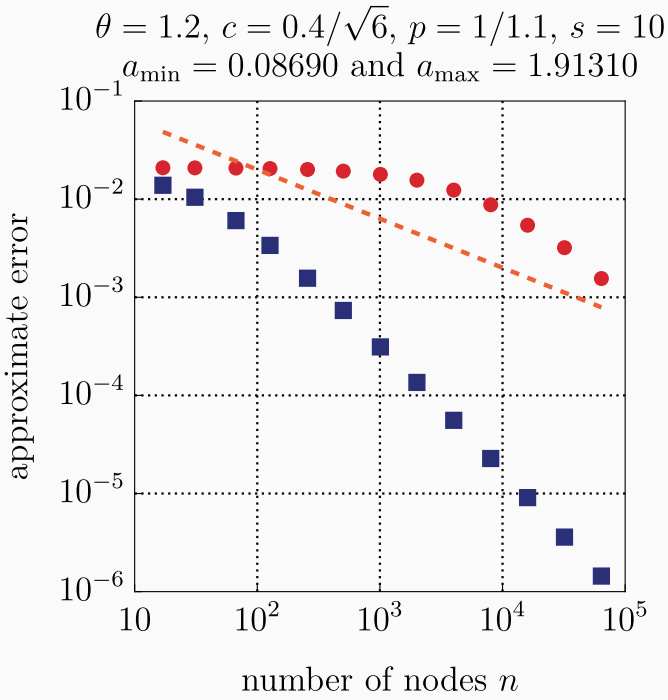
in the physical domain $D = (0,1)^2$ with the diffusion coefficient

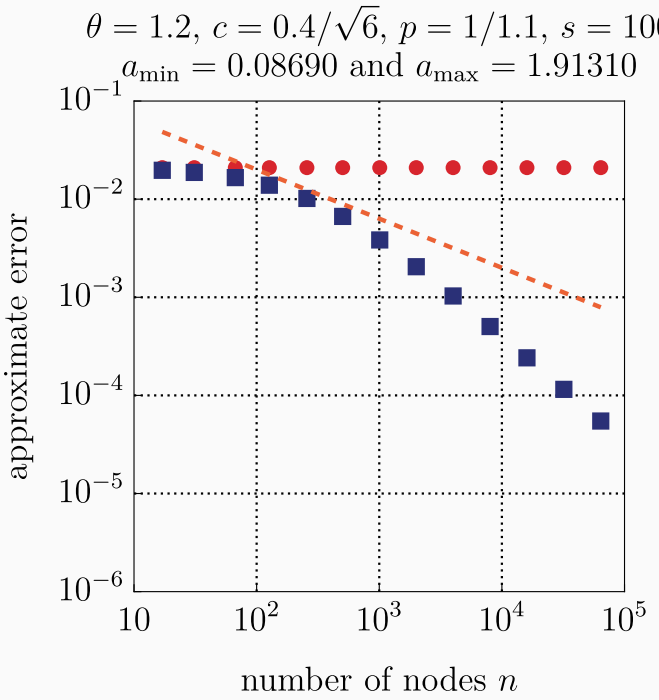
$$a_{\text{per}}(\boldsymbol{x}, \boldsymbol{y}) = 1 + \sum_{j=1}^{100} \sin(2\pi y_j)\psi_j(x), \quad y_j \in [0,1],$$

where $\psi_j(\boldsymbol{x}) = cj^{-\theta} \sin(j\pi x_1) \sin(j\pi x_2)$. **Note that** $\|\psi_j\|_{L^\infty} \propto j^{-\theta}$.

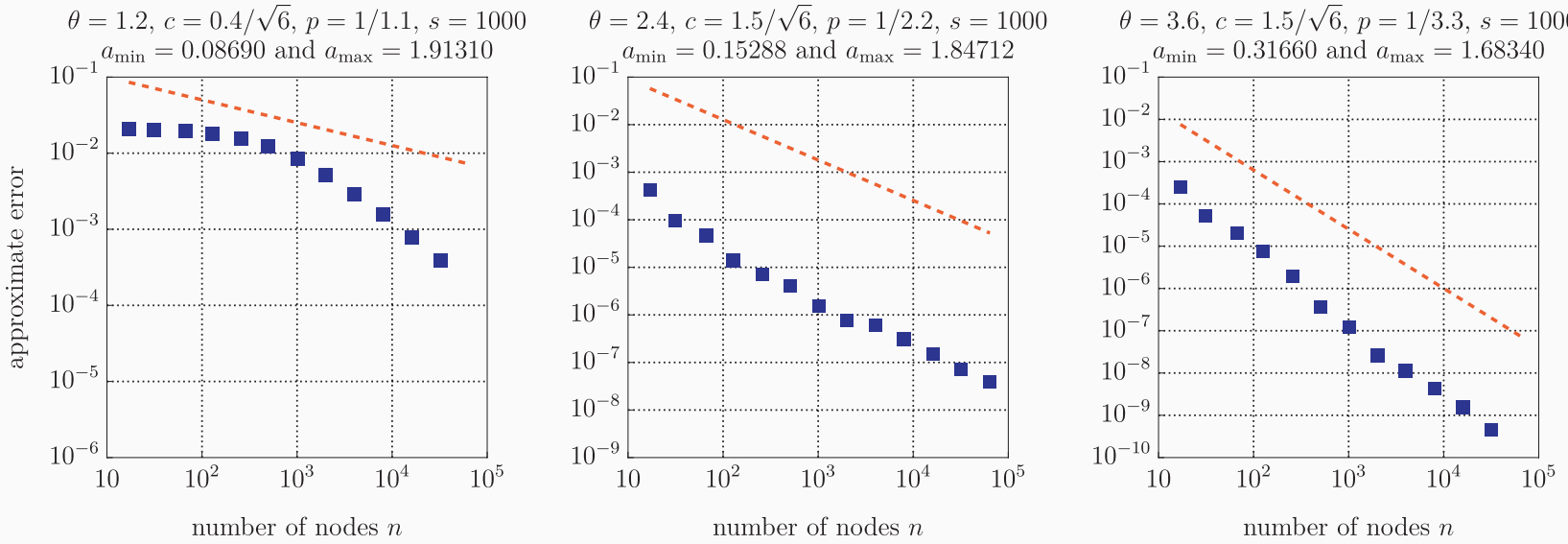


In certain situations, the product weights can outperform SPOD weights.





The product weights can be used to perform computations for higher dimensional problems (here, $s = 1000$).



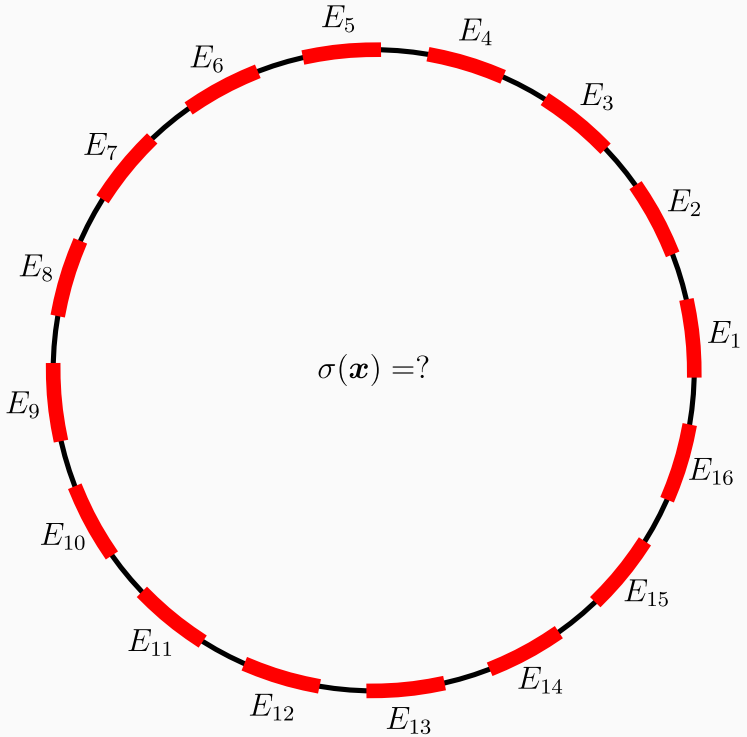
Part IV: Application to an inverse problem

The complete electrode model

Let $D := \{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x}\| \leq 1\}$. Let $\{E_k\}_{k=1}^L \subseteq \partial D$ be an array of $L := 16$ equidistantly spaced non-overlapping electrodes of width 0.2 on the boundary ∂D. Fix the current feed $\boldsymbol{I} \in \mathbb{R}^L_\diamond$ and let $\sigma \in L^\infty_+(D)$. The *forward problem* is to find the electromagnetic potential $u \in H^1(D)$ as well as $\boldsymbol{U} \in \mathbb{R}^L$, the potentials on the electrodes, which satisfy

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 & \text{in } D, \\ \sigma \frac{\partial u}{\partial \boldsymbol{n}} = 0 & \text{on } \partial D \setminus \bigcup_{k=1}^L \overline{E_k}, \\ u + z_k \sigma \frac{\partial u}{\partial \boldsymbol{n}} = U_k & \text{on } E_k,\ k \in \{1, \ldots, L\}, \\ \int_{E_k} \sigma \frac{\partial u}{\partial \boldsymbol{n}} \, \mathrm{d}S = I_k, & k \in \{1, \ldots, L\}, \end{cases}$$

with $\boldsymbol{n}$ denoting the outer normal.
Moreover, we take $z_k = 1 \ \forall k$.



The forward problem is solved numerically using EIDORS software (FEM).

Fix the current pattern $\boldsymbol{I}_k := \boldsymbol{e}_1 - \boldsymbol{e}_{k+1} \in \mathbb{R}^L_\diamond$, $k \in \{1, \dots, L-1\}$.

Kernel-based surrogate for the forward problem: Let us parameterize the conductivity as

$$\sigma(\boldsymbol{x}, \boldsymbol{y}) := 1 + \frac{1}{\sqrt{6}} \sum_{k=1}^{s} \sin(2\pi y_k)\psi_k(\boldsymbol{x}), \quad \boldsymbol{x} \in D, \ \boldsymbol{y} \in [0,1]^s,$$

where $\psi_k(\boldsymbol{x}) := \frac{1}{(i_k^2+j_k^2)^\vartheta} \sin(\pi i_k \frac{x_1+1}{2}) \sin(\pi j_k \frac{x_2+1}{2})$, $\vartheta = 1.2$, the sequence $(i_k, j_k)_{k\geq 1}$ is an ordering of the elements of $\mathbb{N} \times \mathbb{N}$ s.t. $\|\psi_k\|_{L^\infty} = \mathcal{O}(k^{-\vartheta})$ by Weyl's asymptotics. We set $s = 30$.

Denote by $\mathcal{U}(\boldsymbol{y}) := \mathrm{vec}([\boldsymbol{U}_1, \dots, \boldsymbol{U}_{L-1}]) \in \mathbb{R}^{L(L-1)}$ the (flattened) voltage matrix, comprised of the electrode potential measurements corresponding to the current pattern $\boldsymbol{I}_1, \dots, \boldsymbol{I}_{L-1}$ and $\boldsymbol{y} \in [0,1]^s$.

We construct the (vector-valued) QMC–kernel interpolant $\mathcal{U}_n(\boldsymbol{y}) := \sum_{k=1}^n \boldsymbol{c}_k K(\boldsymbol{t}_k, \boldsymbol{y}) \in \mathbb{R}^{L(L-1)}$ (using **serendipitous weights**) for the mapping $G : \boldsymbol{y} \mapsto \mathcal{U}(\boldsymbol{y})$ based on $n = 1\,024\,207$ QMC nodes satisfying $G(\boldsymbol{t}_k) = \mathcal{U}_n(\boldsymbol{t}_k)$ $\forall k$.

Experiment setup

We have constructed the QMC–kernel interpolant $\mathcal{U}_n(\boldsymbol{y})$ offline based on the periodically parameterized model for $\sigma(\boldsymbol{x}, \boldsymbol{y})$. For the numerical experiments, we

- fix some target conductivity σ_{target} and numerically compute the "exact" electrode potential measurements $\boldsymbol{U}_{\text{exact}} := \text{vec}([\boldsymbol{U}^1_{\text{exact}}, \ldots, \boldsymbol{U}^{L-1}_{\text{exact}}]) \in \mathbb{R}^{L(L-1)}$. To avoid the inverse crime, we do not use the same FE mesh that was used to build the surrogate; instead we use a finer FE mesh.
- we contaminate the electrode potential measurements with noise

$$\boldsymbol{U}_{\text{noisy}} = \boldsymbol{U}_{\text{exact}} + \boldsymbol{\eta}, \quad \boldsymbol{\eta} \sim \mathcal{N}(\boldsymbol{0}, \tau^2 I),$$

where $\tau := 10^{-3} \max_{j,k=1,\ldots,L(L-1)} |(U_{\text{exact}})_j - (U_{\text{exact}})_k|$.

Our reconstruction is $\sigma(\boldsymbol{x}, \boldsymbol{y}^*)$, where

$$\boldsymbol{y}^* := \underset{\boldsymbol{y} \in [0,1]^s}{\arg\min} \{\|\boldsymbol{U}_{\text{noisy}} - \mathcal{U}_n(\boldsymbol{y})\|^2\}.$$

The minimization is carried out using `lsqnonlin` in MATLAB with the `levenberg-marquardt` algorithm.

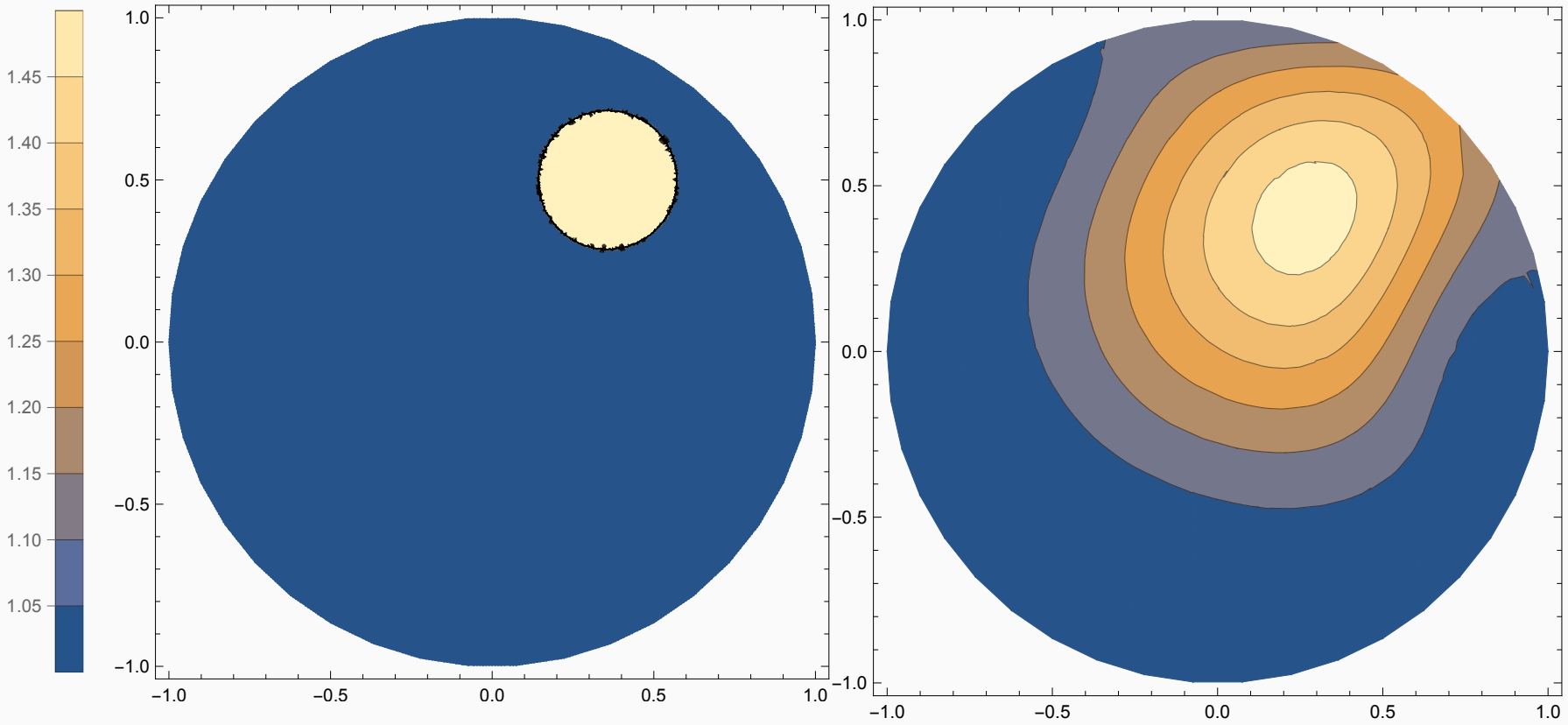


Figure 10: Left: target conductivity. Right: reconstructed conductivity.

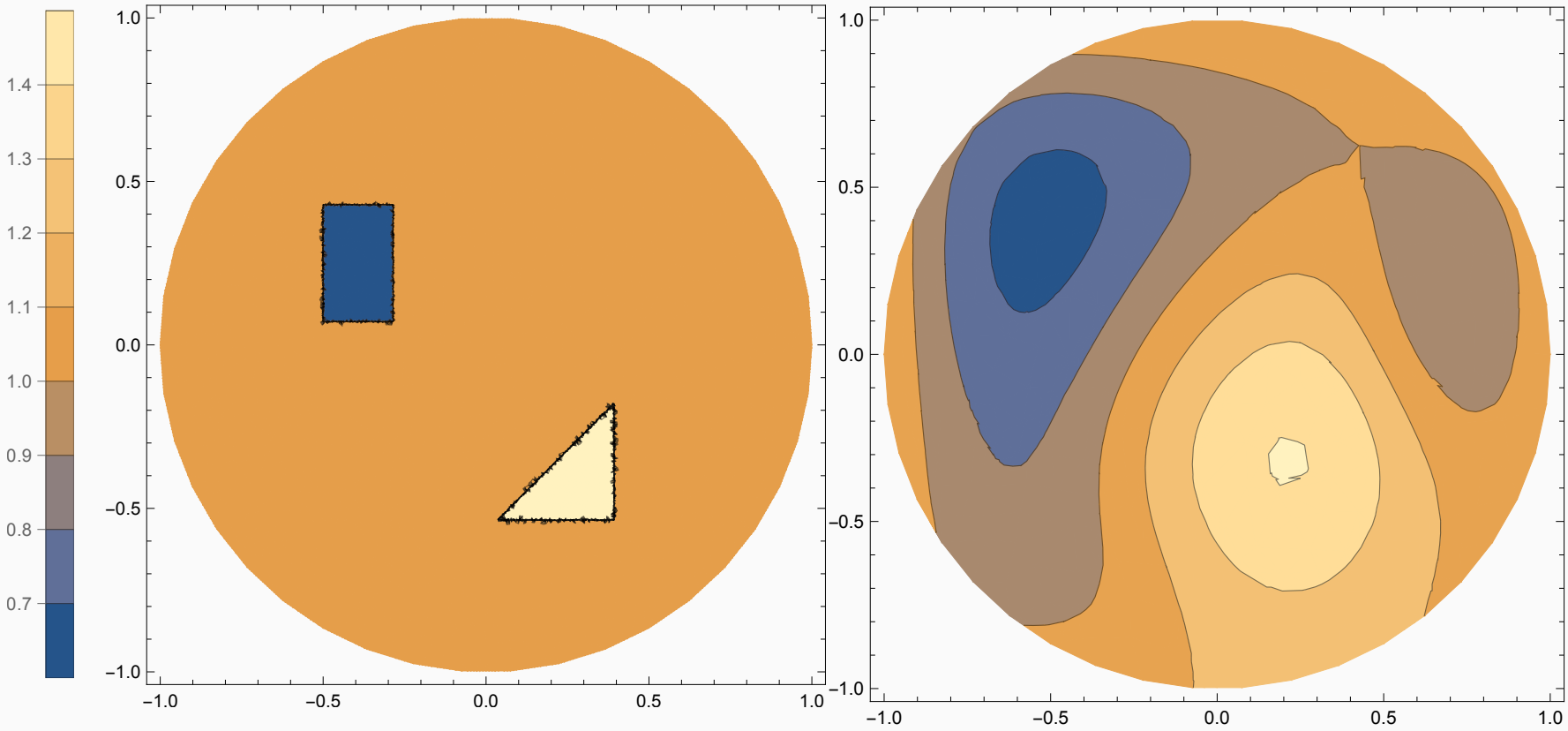


Figure 11: Left: target conductivity. Right: reconstructed conductivity.

Conclusions

- Kernel interpolation method that can be used to approximate the output high-dimensional parametric PDEs. Kernel interpolant can be constructed efficiently at cost $\mathcal{O}(n \log n)$. No multi-index sets! (Compare with sparse grids or trigonometric approximation.)
- Using product weights, practical for challenging high-dimensional problems (e.g., as surrogates for Bayesian inversion).
- For EIT, the kernel interpolation scheme could be useful for efficient recovery of other uncertainties (domain shape, electrode positions, contact resistances, etc.).

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