Infinite Dimensional Nonlocal Diffusions with Additive Noise

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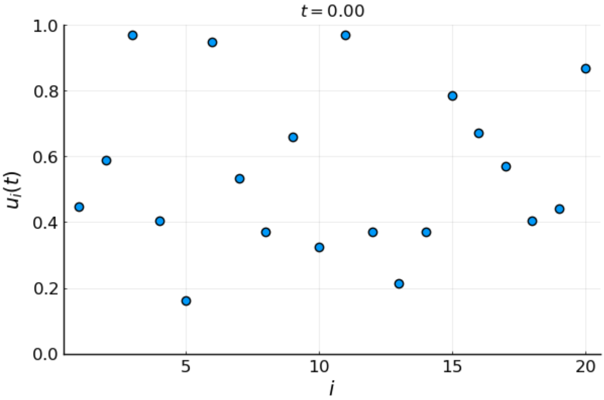
Kuramoto Oscillator Systems

$$\frac{du_i}{dt} = \frac{(-1)^\alpha}{2k+1} \sum_{d(i,j)_{\mathrm{per}} \leqslant k} \sin(2\pi(u_j - u_i)), \quad i = 1, \ldots n \tag{1}$$

- Nonlinearly couples n phase oscillators, $\{u_i\}$
- $k = 1, 2, \ldots$-nearest neighbors (under periodicity) are coupled.
- Exhibits synchronization and other dynamical behavior

Synchronization of Kuramoto



- Random $U(0,1)$ initial data
- Converges to a "twisted" state

Energetic Formulation

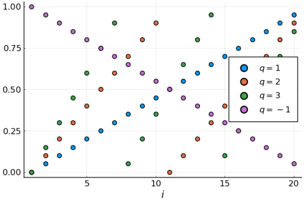
Energy

$$E[u] = \frac{(-1)^{\alpha+1}}{4\pi} \sum_i \sum_{j \in \mathcal{N}_i} \cos(2\pi(u_j - u_i)) \tag{2}$$

Gradient Flow

$$\dot{u}_i = -\frac{1}{|\mathcal{N}_i|} D_{u_i} E[u], \quad \mathcal{N}_i = \text{Neighbors of } i \tag{3}$$

Twisted States



q-twisted states:

$$u_i^{(q)} = q \cdot i/n, \quad i = 1, \ldots, n \tag{4}$$

for $q \in \mathbb{Z}$

- Exact, steady state, solutions
- Variety of stability properties depending on coupling (attractive/repulsive), n, q, interaction range
- See Girnyk et al. (2012), Medvedev & Tang (2015), others for stability studies

Deriving the Continuum Limit

- Let $\mathcal{N}_i = \mathcal{N}_i^{(n)}$ include the $k^{(n)}$ nearest neighbors
- Assume $k^{(n)}/n = r$, fixed, as $n \to \infty$
- Piecewise constant Galerkin fomrulation:

$$u^n(x,t) = \sum_{i=1}^{n} 1_{(x_{i-1},x_i]} u_i^n(t) \tag{5}$$

- At $x \in (x_{i-1}, x_i]$

$$\begin{aligned}
\partial_t u^n &= \sum_i \frac{(-1)^\alpha}{2k^{(n)}+1} \sum_{j \in \mathcal{N}_i^{(n)}} \sin(2\pi(u_j^n - u_i^n)) 1_{(x_{i-1},x_i]} \\
&= (-1)^\alpha \frac{n}{2k^{(n)}+1} \sum_j \frac{1}{n} 1_{d_{\mathrm{per}}(i,j) \leqslant k} \sin(2\pi(u_j^n - u_i^n)) \qquad (6) \\
&\approx \frac{(-1)^\alpha}{2r} \int_0^1 1_{d_{\mathrm{per}}(x,y) \leqslant r} \sin(2\pi(u^n(y) - u^n(x))) dy
\end{aligned}$$

Generalized Continuum Limit

- With $u : I^d \times \mathbb{R}^+ \to \mathbb{R}$,

$$\partial_t u = f(t,u) + \underbrace{\int_{[0,1]^d} K(x,y) S(u(x),u(y)) dy}_{\mathbf{K}[u]} \tag{7}$$

- Interaction kernel K satisfying bounded/intergrability conditions:

$$\text{ess sup}_x \|K(x,\bullet)\|_{L^2} < \infty, \quad \text{ess sup}_y \|K(\bullet,y)\|_{L^2} < \infty \tag{8}$$

- Nonlinear coupling S satisfying Lipschitz/linear growth conditions:

$$|S(u,v)| \leqslant A_S + B_S(|u| + |v|), \tag{9a}$$
$$|S(u,v) - S(u',v')| \leqslant L_S(|u-u'| + |v-v'|). \tag{9b}$$

- Similar assumptions on f

Generalized Continuum Limit, Continued

Assume

- $f = 0$
- $K(x, y) = K(y, x)$
- $S(u, v) = (-1)^{\alpha} \sin(2\pi(v - u))$

then this flow is also a gradient flow

$$\partial_t u = -D_u E[u] \tag{10}$$

with

$$E[u] = \iint K(x, y) \frac{(-1)^{1+\alpha}}{4\pi} \cos(2\pi(u(x) - u(y))) dxdy \tag{11}$$

Generalizes to other S and $f \neq 0$

Nomenclature – Nonlocal Diffusion

- For the discrete model, with $k = 1$,

$$\dot{u}_i^n = (-1)^\alpha \left[\sin(2\pi(u_{i+1}^n - u_i^n)) + \sin(2\pi(u_{i-1}^n - u_i^n))\right] \tag{12}$$

- As $n \to \infty$, expect $u_{i+1}^n - u_i^n$ to be small:

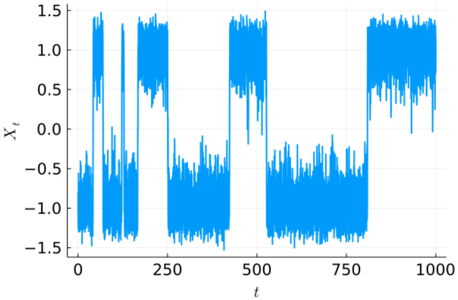
$$\begin{aligned} \dot{u}_i^n &\approx 2\pi(-1)^\alpha \left[(u_{i+1}^n - u_i^n) + (u_{i-1}^n - u_i^n))\right] \\ &\approx 2\pi(-1)^\alpha \Delta x^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \\ &\approx 2\pi(-1)^\alpha \Delta x^2 u_{xx} \end{aligned} \tag{13}$$

Doublewell

- Gradient descent with additive noise:

$$dX_t = -\nabla V(X_t)dt + \sqrt{2\epsilon}dB_t \tag{14}$$

- Consider the case that $V = (x^2-1)^2$ in dimension one



- "Switching" between metastable regions (basins of attraction of V) – expect similar behavior in Kuramoto systems

Challenges & Goals

- Develop a well-posedness theory for the equation with additive noise:

$$du = f(t,u)dt + \left\{ \underbrace{\int K(x,y)S(u(x),u(y))dy}_{\mathbf{K}[u]} \right\} dt + dW^Q \qquad (15)$$

  where W^Q is a Q-Weiner process
- Develop a suitable convergent numerical method
- Address metastability in the problem

Notion of Solutions and Noise

- Working over the Hilbert space $\mathcal{H} = L^2(I^d)$, we define a predictable process u is said to be a mild solution to the IVP if

$$\begin{aligned} u(t) = \xi &+ \int_0^t f(s, u(s))ds \\ &+ \int_0^t \int K(x,y) S(u(x,s), u(y,s)) dydt \\ &+ W^Q(t) \end{aligned} \tag{16}$$

for all $t \in [0, T]$, a.s., with

$$\mathbb{P}\left(\int_0^T \|u(t)\|^2 dt < \infty \right) = 1 \tag{17}$$

- The noise process, W, is a Q-Weiner process, where $W(t) \sim N(0, t\mathbf{Q})$, $\mathbf{Q}$ a positive definite, self-adjoint, trace class operator $(\mathbf{Q} \neq \mathbf{I})$

Main Result

Theorem

Under the assumptions on f, K, and S, for any $\xi \in L^p(\Omega; \mathcal{H})$, $p \geqslant 2$, there exists a $T > 0$ and a unique mild solution such that

$$|||u|||_{p,T} \leqslant C(1 + |||\xi|||_p) \tag{18}$$

where $C = C(T)$, but is independent of ξ.

Norms

- $|||\xi||| = \mathbb{E}[\|\xi\|_{\mathcal{H}}^p]^{1/p}$
- $|||u|||_{p,T} = \text{ess sup}_{t \leqslant T} |||u(t)|||_p$

Corollaries and Remarks on Proof

Corollary

Under the same assumptions as above, we have continuous dependence upon the data, and Hölder-$\frac{1}{2}$ continuity in the $|||\bullet|||_p$ norm.

- Proof is standard, by Banach fixed point method
- **NOTE:** There is no linear term in this problem; no linear semigroup to provide smoothing a la stochastic heat equations:

$$du = \Delta u dt + f(u)dt + dW$$

Semidiscrete Formulation ($d = 1$)

Letting

$$u^n(t,x) = \sum_i u_i^n(t)\chi_i^n(x), \quad \chi_i^n(x) = 1_{(x_{i-1},x_i]} \tag{19}$$

the $\{u_i^n\}$ solve

$$du_i^n = \left\{ f(t, u_i^n) + \Delta x \sum_j K_{ij}^n S(u_i^n, u_j^n) \right\} dt + dW_i^n \tag{20}$$

where

$$K_{i,j}^n = \Delta x^{-2} \iint K(x,y)\chi_i^n(x)\chi_j^n(y)dxdy, \tag{21}$$

$$W_i^n(t) = \Delta x^{-1} \langle W(t,\cdot), \chi_i^n \rangle \tag{22}$$

Semidiscrete Convergence Result

Lemma

$$\left\|\left| u - u^n \right|\right\|_{2,T} \lesssim \left\|(\mathbf{I} - \mathbf{P}_n)g\right\| + \left\|(\mathbf{I} - \mathbf{P}_n^{(1)})K\right\|_{L^2(I^d \times I^d)} + \left\|\left|(\mathbf{I} - \mathbf{P}_n)W\right|\right\|_{2,T}$$

Theorem

Under the additional assumption that S is a strictly bounded, with $u(t = 0) = g \in L^2$,

$$\lim_{n \to \infty} \left\|\left| u - u^n \right|\right\|_{2,T} = 0 \tag{23}$$

- To obtain any sort of rate of convergence, it is necessary to specify additional information, particularly about W

Lipschitz Spaces

Definition

For $\phi \in L^p(I^s)$, $s \in \mathbb{N}$, $p \geqslant 1$,

$$\begin{aligned} \omega_p(\phi, \delta) &= \sup_{|h| \leqslant \delta} \|\phi(\bullet + h) - \phi(\bullet)\|_{L^p(I_h^s \bigcap I^s)}, \ \delta > 0, \\ I_h^s &= \{x \in \mathbb{R}^s : \ x + h \in I^s\}, \end{aligned} \tag{24}$$

is called the L^p-modulus of continuity of ϕ. For $\alpha \in (0, 1]$, the Lipschitz space $\mathrm{Lip}\,(\alpha, L^p(I^s))$ is defined as follows

$$\begin{aligned} \mathrm{Lip}\,(\alpha, L^p(I^s)) &= \{\phi \in L^p(I^s) : \ \exists C > 0 : \ \omega_p(\phi, \delta) \leqslant C\delta^\alpha\}, \\ \|\phi\|_{p,\alpha} &= \limsup_{\delta \to 0} \delta^{-\alpha} \omega_p(\phi, \delta). \end{aligned} \tag{25}$$

Convergence with Rates

Theorem

Let λ_k be the eigenvalues of Q with eigenvectors e_k. Let $g \in \mathrm{Lip}\left(\alpha, L^2(I)\right)$ and $K \in \mathrm{Lip}\left(\beta, L^2(I^2)\right)$ for some $\alpha, \beta \in (0,1]$. Then

$$\|\!| u - u^n |\!\|_{2,T} \lesssim n^{-\alpha} + n^{-\beta} + \underbrace{\sqrt{\inf_{m\in\mathbb{N}} \left\{ \sum_{k=1}^{m} \lambda_k \omega_2(e_k, n^{-1})^2 + \sum_{k=m+1}^{\infty} \lambda_k \right\}}}_{\Psi(n;Q)} \tag{26}$$

NOTE: Need to specify details on covariance Q to obtain a rate

Convergence with Rates, Continued

- Let

$$Q = \left(-\frac{d^2}{dx^2}\right)^{-1} \tag{27}$$

  with Dirichlet boundary conditions on $(0,1)$
- $e_k = \sqrt{2}\sin(k\pi x)$ and $\lambda_k = (\pi k)^{-2}$
- **Claim:**

$$\Psi(n) \lesssim n^{-1/2} \tag{28}$$

  and

$$\|\!| u - u^n \|\!|_{2,T} \lesssim n^{-\alpha} + n^{-\beta} + n^{-1/2} \tag{29}$$

Fully Discrete Formulation ($d = 1$)

- Let $u^{n,k}$ be the approximation of $u^n(t_k)$

$$u^{n,k+1} = u^{n,k} + f(t_k, u^{n,k})\Delta t + \mathbf{K}^n[u^{n,k}]\Delta t + \Delta W^{n,k+1} \tag{30}$$

- The Gaussian process increment corresponds to

$$\Delta W^{n,k+1} = \mathbf{P}^n(W(t_{k+1}) - W(t_k)) = W^{n,k+1} - W^{n,k} \tag{31}$$

- We analyze the error as:

$$\Delta^{n,k} = \||u(t_k) - u^{n,k}\||_2 \leqslant \underbrace{\||u(t_k) - u^k\||_2}_{\equiv \Delta_t^k} + \underbrace{\||u^k - u^{n,k}\||_2}_{\equiv \Delta_x^{n,k}} \tag{32}$$

- Δ_t^k is the error due to the time step error
- $\Delta_x^{n,k}$ is the error due to, only, the spatial error *of the time discretized problem*

Fully Discrete Convergence Result

Theorem

Under the same assumptions as those for the semidiscrete problem

$$\max_k |||u(t_k) - u^{n,k}|||_2 \lesssim n^{-\alpha} + n^{-\beta} + \Psi(n) + \sqrt{\Delta t}$$

- The error due to spatial discretization is uniform over all Δt small enough
- The $O(\sqrt{\Delta t})$ error is the strong error of Euler-Maruyama on a Banach space valued SDE
- We can improve this to $O(\Delta t)$

Test Problem Setup

- $f = 0$
- $S = \sin(2\pi(u - v))$
- For K with $r = 0.3$

$$A_r = \{(x, y) \in [0, 1]^2 \mid \min\{|x - y|, 1 - |x - y|\} < r\}$$
$$K(x, y) = 1_{A_r}(x, y)$$

- Take $\mathbf{Q} = (-d^2/dx^2)^{-s/2}$ with periodic boundary conditions, and vary $s > 1$
- Initial condition is $u_0 = x(1 - x)$

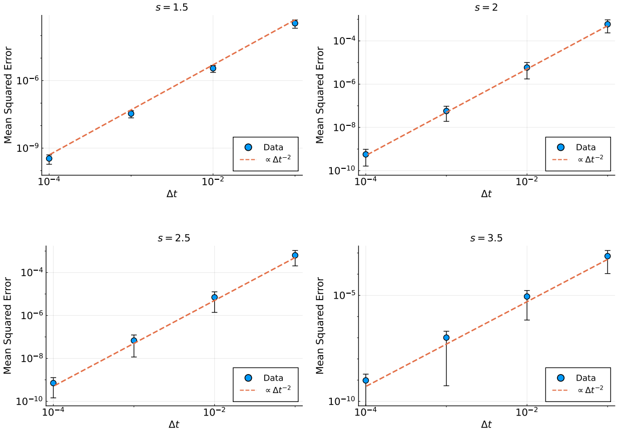
Test Problem Setup, Continued

- Compare against a reference solution at $T = 10$

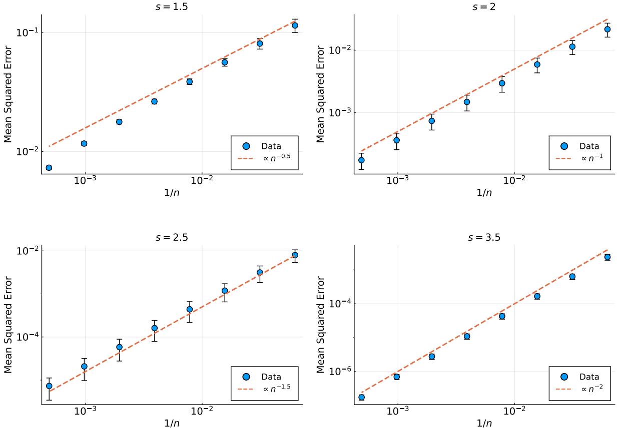
$$\begin{aligned} \mathsf{MSE} &= \mathbb{E}[\|u_{\mathrm{ref}}(T) - u_{\Delta x, \Delta t}(T)\|^2] \\ &\lesssim n^{-2} + n^{-1} + \begin{cases} n^{-(s-1)} & s \neq 3 \\ n^{-2} \log n & s = 3 \end{cases} \end{aligned} \tag{33}$$

- For rough noise (small s), this is dominated by $n^{-(s-1)}$
- For sufficiently smooth noise, $s > 2$, this is dominated by n^{-1} from the Galerkin approximation of $\mathbf{K}$ (not quite)
- Reference solution is computed with a $\Delta x_{\mathrm{ref}} \ll \Delta x$ and/or $\Delta t_{\mathrm{ref}} \ll \Delta t$

Fixed Spatial Discretization, Varying Time Step



Fixed Time Step, Varying Spatial Discretization



$s = 2.5$ should be n^{-1}, dominated by the projection error of $\mathbf{K}$

Improved Convergence

- We analyzed the error term

$$\left\|\left|\mathbf{K}[u^n] - \mathbf{K}^n[u^n]\right|\right\|_2 \lesssim \|(\mathbf{I} - \mathbf{P}_n^{(1)})\mathbf{K}\|_{L^2_{xy}} \lesssim n^{-1/2} \tag{34}$$

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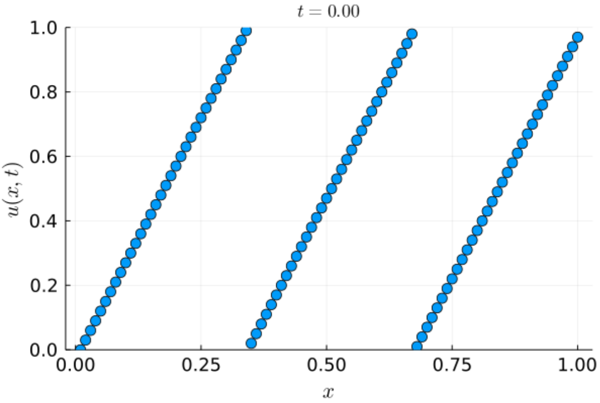
- Instead, we could have looked at anistropic norms

$$\|\|\mathbf{K}[u^n] - \mathbf{K}^n[u^n]\|\|_2 \lesssim \|(\mathbf{I} - \mathbf{P}_n^{(1)})\mathbf{K}\|_{L^1_y L^\infty_x} \lesssim n^{-1} \tag{35}$$

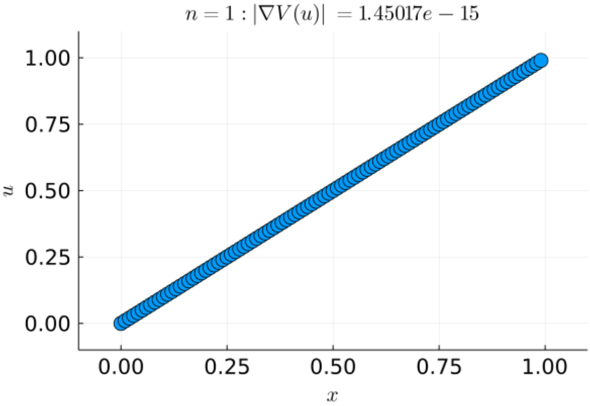
- This holds for set function type $K(x, y) = 1_A(x, y)$

Noise Driven Transition



String Method Analysis

$n = 100$, $k = 15$, attractive coupling



Remarks

- Stochastic continuum limit of Kuramoto is a well posed problem
- We have formulated a convergence numerical method with strong convergence depending on regularity of $\mathbf{K}$ and the noise process
- Numerical experiments reveal noise induced transitions. Ongoing efforts to study the problem by Freidlin-Wentzell and string method and identify new stationary states (saddles)
- Outstanding challenges:
  1. Multiplicative noise
  2. Space-time white noise?
  3. Expand class of interaction functions for $O(\Delta t)$
  4. Develop a refined framework for projection of $\mathbf{K}$

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