The interpolated drift implicit Euler MLMC method for pricing Barrier options and applications to the CIR and CEV models

Ahmed Kebaier
Université d'Évry

(joint work with Mouna Ben Derouich)

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Outline of The Talk

1. The Euler Multilevel Monte Carlo scheme

Motivation

- We are interested in approximating barrier option prices such as the Down-and-Out (D-O) and the Up-and-Out (U-O) barrier options

$$\pi_{\mathcal{B}_D} = \mathbb{E}\Big[f(X_T)\mathbf{1}_{\{\inf_{t\in[0,T]} X_t > \mathcal{B}_D\}}\Big] \text{ and } \pi_{\mathcal{B}_U} = \mathbb{E}\Big[f(X_T)\mathbf{1}_{\{\sup_{t\in[0,T]} X_t < \mathcal{B}_U\}}\Big]$$

for a process $(X_t)_{t\in[0,T]}$ solution to

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \tag{1}$$

where $(W_t)_{t\geq 0}$ is a s.B.M. $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}_+^*$ are loc. Lipschitz-functions such that $\frac{1}{\sigma}$ is loc. integrable.

- For $\phi(y) = \int_{y_0}^{y} \frac{1}{\sigma(x)}dx$, if $\sigma \in \mathcal{C}^1$ then by the Lamperti transform $Y_t = \phi(X_t)$ solves the stochastic differential equation

$$dY_t = L(X_t)dt + dW_t, \quad Y_0 = \phi(x),$$

with $L(x) = \left(\frac{b}{\sigma} - \frac{\sigma'}{2}\right)(\phi^{-1}(x))$.

- As the function ϕ is monotonic, we get $\pi_{\mathcal{B}_D} = \pi_{\mathcal{D}}$ and $\pi_{\mathcal{B}_U} = \pi_{\mathcal{U}}$ where

$$\pi_{\mathcal{D}} = \mathbb{E}\Big[g(Y_T)\mathbf{1}_{\{\inf_{t\in[0,T]} Y_t > \mathcal{D}\}}\Big], \ \pi_{\mathcal{U}} = \mathbb{E}\Big[g(Y_T)\mathbf{1}_{\{\sup_{t\in[0,T]} Y_t < \mathcal{U}\}}\Big],$$

$g(x) = f \circ \phi^{-1}(x)$, $\mathcal{D} = \phi(\mathcal{B}_D)$ and $\mathcal{U} = \phi(\mathcal{B}_U)$.

General Framework

- In the sequel, we consider the general setting given in [Alfonsi 2013] and let $(Y_t)_{t\geq 0}$ denote the SDE defined on $I = (0, +\infty)$ solution to

$$dY_t = L(Y_t)dt + \gamma dW_t, \quad t \geq 0, \; Y_0 = y \in I, \text{ with } \gamma \in \mathbb{R}^*, \tag{2}$$

where $L : I \longrightarrow \mathbb{R}$ *is* C^2, s.t.

$$\exists\, \kappa > 0, \;\; \forall y, y' \in I, y \leq y', L(y') - L(y) \leq \kappa(y' - y). \tag{3}$$

In addition, for an arbitrary point $d \in I$, we assume that

$$v(x) = \int_d^x \int_d^y \exp\Big(- \frac{2}{\gamma^2} \int_z^y L(\xi)d\xi \Big) dzdy \text{ satisfies } \lim_{x \to 0^+} v(x) = +\infty. \tag{H1}$$

- By the Feller's test (3) and (H1) ensure that the SDE (2) admits a unique strong solution $(Y_t)_{t\geq 0}$ on $(0, +\infty)$.

The drift implicit Euler scheme

- For $t_i = \frac{iT}{n}$, $0 \le i \le n$, we consider the drift implicit continuous scheme introduced in [Alfonsi 2013] ,

$$\begin{aligned} \widehat{Y}^n_t &= \widehat{Y}^n_{t_i} + L(\widehat{Y}^n_t)(t - t_i) + \gamma(W_t - W_{t_i}), t \in [t_i, t_{i+1}] \\ \widehat{Y}^n_0 &= y \end{aligned} \tag{4}$$

is well defined and for all $t \in [0, T]$, $\hat{Y}^n_t \in I = (0, +\infty)$.

- If in addition we assume that for $p \ge 1$, we have

$$\mathbb{E}\Big[\Big(\int_0^T |L'(Y_u)L(Y_u) + \frac{\gamma^2}{2}L''(Y_u)|du\Big)^p\Big] < \infty \quad and \quad \mathbb{E}\Big[\Big(\int_0^T (L'(Y_u))^2 du\Big)^{\frac{p}{2}}\Big] < \infty, \tag{H2}$$

then by [Alfonsi 2013], there exists a positive constant K_p such that

$$\mathbb{E}^{\frac{1}{p}}\left[\sup_{t\in[0,T]} |\widehat{Y}^n_t - Y_t|^p\right] \le K_p \frac{T}{n}.$$

- For our purpose, we rather focus on a slightly different interpolated version. For $t_i = \frac{iT}{n}$, $0 \leq i \leq n$,

$$\begin{cases} \overline{Y}^n_{t_{i+1}} & = & \overline{Y}^n_{t_i} + L(\overline{Y}^n_{t_{i+1}})\frac{T}{n} + \gamma(W_{t_{i+1}} - W_{t_i}), \\ \overline{Y}^n_0 & = & y. \end{cases} \tag{5}$$

and then introduce the following interpolated drift implicit scheme

$$\overline{Y}^n_t = \overline{Y}^n_{t_i} + L(\overline{Y}^n_{t_{i+1}})(t - t_i) + \gamma(W_t - W_{t_i}), \quad \text{for } t \in [t_i, t_{i+1}[. \tag{6}$$

- The main advantages of this Brownian interpolation is that it preserves the rate of strong convergence of the original drift implicit scheme (4) and allows an easy use of of the Brownian bridge technique for pricing Barrier options.

Strong convergence rate

For this aim, we strengthen our assumption on L as follows:

$$L : I \to \mathbb{R} \text{ is } \mathcal{C}^2 \text{ such that: } L \text{ is decreasing on } (0, A) \text{ for } A > 0,$$
$$\text{and } L' \text{ satisfies } \exists L'_A > 0 \text{ s.t. } \forall y \in (A, \infty),\ |L'(y)| \leq L'_A. \quad \text{(H3)}$$

Theorem 1

Assume that conditions (H2) and (H3) hold true for a given $p > 1$ and with $L'_A < \frac{n}{2T}$. Then, there exists a constant $K_p > 0$ such that

$$\mathbb{E}^{\frac{1}{p}}\left[\sup_{t\in[0,T]} |\overline{Y}^n_t - Y_t|^p\right] \leq K_p \frac{T}{n}.$$

Corollary 2

Under assumptions of Theorem 1, if in addition

$$\exists\, \alpha > 0 \text{ such that } \forall\, y \in I,\ \ yL(y) \leq \alpha(1 + |y|^2) \qquad \text{(H4)}$$

then $\mathbb{E}[\sup_{0\leq t\leq T} |\overline{Y}^n_t|^p] < \infty$.

Brownian bridge and drift implicit scheme

- The above barrier option prices can be approximated by

$$\overline{\pi}_{\mathcal{D}} := \mathbb{E}\Big[g(\overline{Y}^n_T)\prod_{i=0}^{n-1}\mathbf{1}_{\{\inf_{t\in[t_i,t_{i+1}]}\overline{Y}^n_t>\mathcal{D}\}}\Big] \text{ and } \overline{\pi}_{\mathcal{U}} := \mathbb{E}\Big[g(\overline{Y}^n_T)\prod_{i=0}^{n-1}\mathbf{1}_{\{\sup_{t\in[t_i,t_{i+1}]}\overline{Y}^n_t<\mathcal{U}\}}\Big].$$

- To get more accurate approximations, we use the Brownian bridge technique. For $x \in \mathbb{R}$, $(x)_+$ stands for $\max(x, 0)$.

Proposition 1

Under the above notation, for $h = \frac{T}{n}$, we have

$$\overline{\pi}_{\mathcal{D}} = \mathbb{E}\Big[g(\overline{Y}^n_T)\prod_{i=0}^{n-1}(1-\overline{q}_i)\Big], \overline{q}_i := \exp\left(\frac{-2(\overline{Y}^n_{t_i}-\mathcal{D})_+(\overline{Y}^n_{t_{i+1}}-\mathcal{D})_+}{\gamma^2 h}\right)$$

and

$$\overline{\pi}_{U} = \mathbb{E}\Big[g(\overline{Y}^n_T)\prod_{i=0}^{n-1}(1-\overline{p}_i)\Big], \overline{p}_i = \exp\left(\frac{-2(\mathcal{U}-\overline{Y}^n_{t_i})_+(\mathcal{U}-\overline{Y}^n_{t_{i+1}})_+}{\gamma^2 h}\right).$$

The Brownian bridge technic goes back to [Baldi 1995] and [Gobet 2009] for related refinements.

The interpolated drift implicit Euler MLMC method

- We consider the drift implicit scheme $(\overline{Y}^{2^\ell}_{t_i})_{0\le i\le 2^\ell}$ given in (5) using a time step $h_\ell = 2^{-\ell}T$ for $\ell \in \{0, \dots, L\}$, with $L = \log n/\log 2$, where n is the finest time step number.

- Let $(\overline{Y}^{2^\ell}_t)_{0\le t\le T}$ denote the Brownian interpolation of the drift implicit scheme defined in (6) with time step h_ℓ. For

$$\overline{P}_\ell := g(\overline{Y}^{2^\ell}_T)\prod_{i=0}^{2^\ell-1}\mathbf{1}_{\{\sup_{t\in[t_i^\ell,t_{i+1}^\ell]}\overline{Y}^{2^\ell}_t<\mathcal{U}\}}, \qquad \text{where } t_i^\ell = \frac{iT}{2^\ell} \quad \text{for } \ell\in\{0,\dots,L\}, \tag{7}$$

we have

$$\overline{\pi}_{\mathcal{U}} = \mathbb{E}\big[\overline{P}_L\big] = \mathbb{E}\big[\overline{P}_0\big] + \sum_{\ell=1}^{L}\mathbb{E}\big[\overline{P}_\ell - \overline{P}_{\ell-1}\big], \tag{8}$$

where $\overline{\pi}_{\mathcal{U}} := \mathbb{E}\Big[g(\overline{Y}^n_T)\prod_{i=0}^{n-1}\mathbf{1}_{\{\sup_{t\in[t_i,t_{i+1}]}\overline{Y}^n_t<\mathcal{U}\}}\Big]$.

- Applying Proposition 1 yields

$$\mathbb{E}\left[\overline{P}_\ell\right] = \mathbb{E}\left[\overline{P}^f_\ell\right], \text{ where } \overline{P}^f_\ell := g(\overline{Y}^{2^\ell}_T) \prod_{i=0}^{2^\ell-1} (1 - \overline{p}^{2^\ell}_i) \text{ with} \tag{9}$$

$$\overline{p}^{2^\ell}_i = \exp\left(\frac{-2(\mathcal{U} - \overline{Y}^{2^\ell}_{t^\ell_i})_+(\mathcal{U} - \overline{Y}^{2^\ell}_{t^\ell_{i+1}})_+}{\gamma^2 h_\ell}\right).$$

- Now, following the conditional MC proposed by [Giles et al. 2019] we get

$$\mathbb{E}[\overline{P}_{\ell-1}] = \mathbb{E}[g(\overline{Y}^{2^{\ell-1}}_T) \prod_{i=0}^{2^{\ell-1}-1} \mathbb{E}[\mathbf{1}_{\{\sup_{t\in[t^{\ell-1}_i, t^{\ell-1}_{i+1}]} \overline{Y}^{2^{\ell-1}}_t < \mathcal{U}\}} | \overline{Y}^{2^{\ell-1}}_{t^{\ell-1}_i}, \overline{Y}^{2^{\ell-1}}_{t^\ell_{2i+1}}, \overline{Y}^{2^{\ell-1}}_{t^{\ell-1}_{i+1}}]] =$$

$$\mathbb{E}[g(\overline{Y}^{2^{\ell-1}}_T) \prod_{i=0}^{2^{\ell-1}-1} \mathbb{E}[\mathbf{1}_{\{\sup_{t\in[t^{\ell-1}_i, t^\ell_{2i+1}]} \overline{Y}^{2^{\ell-1}}_t < \mathcal{U}\}} \mathbf{1}_{\{\sup_{t\in[t^\ell_{2i+1}, t^{\ell-1}_{i+1}]} \overline{Y}^{2^{\ell-1}}_t < \mathcal{U}\}} | \overline{Y}^{2^{\ell-1}}_{t^{\ell-1}_i}, \overline{Y}^{2^{\ell-1}}_{t^\ell_{2i+1}}, \overline{Y}^{2^{\ell-1}}_{t^{\ell-1}_{i+1}}]],$$

where the coarse scheme $\overline{Y}^{2^{\ell-1}}_{t^\ell_{2i+1}}$ is computed using our Brownian interpolation scheme (6) that is

$$\overline{Y}^{2^{\ell-1}}_{t^\ell_{2i+1}} = \overline{Y}^{2^{\ell-1}}_{t^{\ell-1}_i} + L(\overline{Y}^{2^{\ell-1}}_{t^{\ell-1}_{i+1}})(t^\ell_{2i+1} - t^{\ell-1}_i) + \gamma(W_{t^\ell_{2i+1}} - W_{t^{\ell-1}_i}).$$

Brownian bridge for the MLMC method (continued)

- Thus, we rewrite $\sup_{t\in[t_i^{\ell-1},t_{2i+1}^{\ell}]}\overline{Y}_t^{2^{\ell-1}}$ and $\sup_{t\in[t_{2i+1}^{\ell},t_{i+1}^{\ell-1}]}\overline{Y}_t^{2^{\ell-1}}$ as follows

$$\sup_{t\in[t_i^{\ell-1},t_{2i+1}^{\ell}]}\overline{Y}_t^{2^{\ell-1}} = \overline{Y}_{t_i^{\ell-1}}^{2^{\ell-1}} + \gamma \sup_{t\in[t_i^{\ell-1},t_{2i+1}^{\ell}]}\left(W_t - W_{t_i^{\ell-1}} + \frac{1}{\gamma}L(\overline{Y}_{t_{i+1}^{\ell-1}}^{2^{\ell-1}})(t - t_i^{\ell-1})\right),$$

$$W_{t_{2i+1}^{\ell}} - W_{t_i^{\ell-1}} + \frac{1}{\gamma}L(\overline{Y}_{t_{i+1}^{\ell}}^{2^{\ell-1}})(t_{2i+1}^{\ell} - t_i^{\ell-1}) = \frac{1}{\gamma}\left(\overline{Y}_{t_{2i+1}^{\ell}}^{2^{\ell-1}} - \overline{Y}_{t_i^{\ell-1}}^{2^{\ell-1}}\right)$$

and

$$\sup_{t\in[t_{2i+1}^{\ell},t_{i+1}^{\ell-1}]}\overline{Y}_t^{2^{\ell-1}} = \overline{Y}_{t_{2i+1}^{\ell}}^{2^{\ell-1}} + \gamma \sup_{t\in[t_i^{\ell-1},t_{2i+1}^{\ell}]}\left(W_t - W_{t_{2i+1}^{\ell}} + \frac{1}{\gamma}L(\overline{Y}_{t_{i+1}^{\ell-1}}^{2^{\ell-1}})(t - t_{2i+1}^{\ell})\right),$$

$$W_{t_{i+1}^{\ell-1}} - W_{t_{2i+1}^{\ell}} + \frac{1}{\gamma}L(\overline{Y}_{t_{i+1}^{\ell-1}}^{2^{\ell-1}})(t_{i+1}^{\ell-1} - t_{2i+1}^{\ell}) = \frac{1}{\gamma}\left(\overline{Y}_{t_{i+1}^{\ell-1}}^{2^{\ell-1}} - \overline{Y}_{t_{2i+1}^{\ell}}^{2^{\ell-1}}\right).$$

Then, using the Girsanov theorem, we get

$$\mathbb{E}\left[\overline{P}_{\ell-1}\right] = \mathbb{E}\left[\overline{P}^c_{\ell-1}\right], \text{ where } \overline{P}^c_{\ell-1} := g(\overline{Y}^{2^{\ell-1}}_T) \prod_{i=0}^{2^{\ell-1}-1} (1-\overline{p}^{2^{\ell-1}}_{i,1})(1-\overline{p}^{2^{\ell-1}}_{i,2})$$

with

$$\begin{aligned}\overline{p}^{2^{\ell-1}}_{i,1} &= \exp\left(\frac{-2(\mathcal{U}-\overline{Y}^{2^{\ell-1}}_{t^{\ell-1}_i})_+(\mathcal{U}-\overline{Y}^{2^{\ell-1}}_{t^{\ell}_{2i+1}})_+}{\gamma^2 h_\ell}\right),\\ \overline{p}^{2^{\ell-1}}_{i,2} &= \exp\left(\frac{-2(\mathcal{U}-\overline{Y}^{2^{\ell-1}}_{t^{\ell}_{2i+1}})_+(\mathcal{U}-\overline{Y}^{2^{\ell-1}}_{t^{\ell-1}_{i+1}})_+}{\gamma^2 h_\ell}\right),\end{aligned}$$

which can be rewritten as

$$\overline{P}^c_{\ell-1} := g(\overline{Y}^{2^{\ell-1}}_T) \prod_{i=0}^{2^\ell-1} (1-\overline{p}^{2^{\ell-1}}_i) \text{ with } \overline{p}^{2^{\ell-1}}_i = \exp\left(\frac{-2(\mathcal{U}-\overline{Y}^{2^{\ell-1}}_{t^\ell_i})_+(\mathcal{U}-\overline{Y}^{2^{\ell-1}}_{t^\ell_{i+1}})_+}{\gamma^2 h_\ell}\right), \tag{10}$$

where the coarse scheme $\overline{Y}^{2^{\ell-1}}_{t^\ell_i}$ evaluated over the finest time grid is computed using the Brownian interpolation scheme (6).

Brownian bridge for the MLMC method (continued)

- Thus, the improved MLMC method approximates $\overline{\pi}_{\mathcal{U}}$ by

$$\bar{P}_{\mathcal{U}} := \frac{1}{N_0}\sum_{k=1}^{N_0}\overline{P}^f_{0,k} + \sum_{\ell=1}^{L}\frac{1}{N_\ell}\sum_{k=1}^{N_\ell}\left(\overline{P}^f_{\ell,k} - \overline{P}^c_{\ell-1,k}\right), \qquad (11)$$

where the condition $\mathbb{E}\left[\overline{P}^f_{\ell-1}\right] = \mathbb{E}\left[\overline{P}^c_{\ell-1}\right]$ is satisfied.

- Similarly, the improved MLMC method approximates $\overline{\pi}_{\mathcal{D}}$ by

$$\bar{Q}_{\mathcal{D}} := \frac{1}{N_0}\sum_{k=1}^{N_0}\overline{Q}^f_{0,k} + \sum_{\ell=1}^{L}\frac{1}{N_\ell}\sum_{k=1}^{N_\ell}\left(\overline{Q}^f_{\ell,k} - \overline{Q}^c_{\ell-1,k}\right), \qquad (12)$$

where

$$\overline{Q}^f_\ell := g(\overline{Y}^{2^\ell}_T)\prod_{i=0}^{2^\ell-1}(1-\overline{q}^{2^\ell}_i) \text{ with } \overline{q}^{2^\ell}_i = \exp\left(\frac{-2(\overline{Y}^{2^\ell}_{t^\ell_i}-\mathcal{D})_+(\overline{Y}^{2^\ell}_{t^\ell_{i+1}}-\mathcal{D})_+}{\gamma^2 h_\ell}\right)$$

$$\overline{Q}^c_{\ell-1} := g(\overline{Y}^{2^{\ell-1}}_T)\prod_{i=0}^{2^\ell-1}(1-\overline{q}^{2^{\ell-1}}_i) \text{ with } \overline{q}^{2^{\ell-1}}_i = \exp\left(\frac{-2(\overline{Y}^{2^{\ell-1}}_{t^\ell_i}-\mathcal{D})_+(\overline{Y}^{2^{\ell-1}}_{t^\ell_{i+1}}-\mathcal{D})_+}{\gamma^2 h_\ell}\right)$$

Extreme path events

For $p \geq 1$, assumption (H2) is valid and $\sup_{t \in [0,T]} \mathbb{E}\left[|L(Y_t)|^p\right] < \infty$. $\qquad (\tilde{\mathrm{H}}2)$

Lemma 3

Assume that conditions $(\tilde{\mathrm{H}}2)$, (H3) *and* (H4) *are satisfied for a given* $p > 1$ *and* $0 < L'_A < \frac{1}{2h_\ell}$, *with* $h_\ell = 2^{-\ell}T$ *sufficiently small. Let* $\eta \in (0,1)$, *then*

$$\mathbb{P}\left(\max\left(\sup_{0 \leq i \leq 2^\ell} (|Y_{t_i^\ell}|, |\overline{Y}^{2^\ell}_{t_i^\ell}|, |\overline{Y}^{2^{\ell-1}}_{t_i^\ell}|)\right) > h_\ell^{-\eta}\right) = o(h_\ell^q)$$

$$\mathbb{P}\left(\max\left(\sup_{0 \leq i \leq 2^\ell} (|Y_{t_i^\ell} - \overline{Y}^{2^\ell}_{t_i^\ell}|, |Y_{t_i^\ell} - \overline{Y}^{2^{\ell-1}}_{t_i^\ell}|, |\overline{Y}^{2^\ell}_{t_i^\ell} - \overline{Y}^{2^{\ell-1}}_{t_i^\ell}|)\right) > h_\ell^{1-\eta}\right) = o(h_\ell^q)$$

$$\sup_{0 \leq i \leq 2^\ell} \mathbb{P}\left(\int_{t_i^\ell}^{t_{i+1}^\ell} |L(Y_s)| ds > h_\ell^{1-\eta}\right) = o(h_\ell^q) \quad \text{for all } 0 < q < p\eta, \text{ and}$$

$$\sup_{0 \leq i \leq 2^\ell} \mathbb{P}\left(\sup_{t \in [t_i^\ell, t_{i+1}^\ell]} |W_t - W_{t_i^\ell}| > h_\ell^{\frac{1}{2}-\eta}\right) = o(h_\ell^q), \quad \text{for all } q > 0.$$

Theorem 4

Let g denote a payoff function satisfying : $\exists C > 0$ s.t. $\forall x, y > 0$ and $\nu \in \mathbb{R}_+$

$$|g(x) - g(y)| \leq C|x - y|(1 + |x|^\nu + |y|^\nu) \text{ and } |g(x)| \leq C(1 + |x|^{\nu+1}). \tag{13}$$

Moreover, assume that conditions $(\tilde{\mathrm{H}}2)$, (H3) and (H4) are satisfied for $p > \frac{14(1+\delta)^2+4(1+\delta)\nu}{\frac{1}{2}-\delta}$, with $\delta \in (0, 1/2)$ and $0 < L'_A < \frac{1}{2h_\ell}$, with $h_\ell = 2^{-\ell}T$ sufficiently small.

If in addition $\inf_{t\in[0,T]} Y_t$ (resp. $\sup_{t\in[0,T]} Y_t$) has a bounded density in the neighborhood of the barrier $\mathcal{D}$ (resp. $\mathcal{U}$), then the MLMC estimator $\bar{Q}_{\mathcal{D}}$ given by (12) (resp. $\bar{P}_{\mathcal{U}}$ given by (11)) for the D-O (resp. U-O) barrier option satisfies

$$\mathrm{Var}(\overline{\mathrm{Q}}^{\mathrm{f}}_\ell - \overline{\mathrm{Q}}^{\mathrm{c}}_\ell) = \mathrm{O}(\mathrm{h}_\ell^{1+\delta}) \ \ (\mathrm{resp.}\mathrm{Var}(\overline{\mathrm{P}}^{\mathrm{f}}_\ell - \overline{\mathrm{P}}^{\mathrm{c}}_\ell) = \mathrm{O}(\mathrm{h}_\ell^{1+\delta})).$$

Time complexity analysis

- Combining the complexity theorem in [Giles 2008] with the above result, we deduce that for any $\delta \in (0, \frac{1}{2})$ the MLMC estimators $\bar{Q}_{\mathcal{D}}$ and $\bar{P}_{\mathcal{U}}$ reach the optimal time complexity $O(\varepsilon^{-2})$, for a given precision $\varepsilon > 0$, and behave like an unbiased Monte Carlo estimator.

- Taking δ close to $\frac{1}{2}$ achieves a smaller variance of the difference between the finer and coarse approximations which is of order $O(h_\ell^\beta)$ with β close to $\frac{3}{2}$ similar to the case of diffusion with Lipschitz coefficients studied in [Giles et al. 2019], but clearly leads to very restrictive conditions on the finiteness of the moments of $(Y_t)_{t\in[0,T]}$ and $(\bar{Y}_t^n)_{t\in[0,T]}$.

Sketch of the proof

- **First event** A_1We consider any of the extreme path events given in Lemma 3. By Cauchy-Schwarz inequality we get

$$\mathbb{E}[(\overline{Q}_\ell^f - \overline{Q}_\ell^c)^2 \mathbf{1}_{A_1}] \leq 2\sqrt{2}\left(\mathbb{E}^{\frac{1}{2}}[(\overline{Q}_\ell^f)^4] + \mathbb{E}^{\frac{1}{2}}[(\overline{Q}_\ell^c)^4]\right)\sqrt{\mathbb{P}[A_1]}.$$

Then, we use Lemma 3 to get that

$$\mathbb{E}[(\overline{Q}_\ell^f - \overline{Q}_\ell^c)^2 \mathbf{1}_{A_1}] = o(h^{\frac{q}{2}}) \text{ for all } 0 < q < p\eta.$$

- **Second event** A_2 corresponds to the non-extreme paths satisfying

$$|\inf_{t\in[0,T]} Y_t - \mathcal{D}| > h_\ell{}^{\frac{1}{2}-\eta(1+\varepsilon)} \text{ for } \eta \in (0, 1/2(1+\varepsilon)) \text{ with } \varepsilon > 0.$$

➞ We prove that for h_ℓ sufficiently small $\prod_{i=0}^{2^\ell-1}(1-\overline{q}_i^{2^\ell})$ and $\prod_{i=0}^{2^\ell-1}(1-\overline{q}_i^{2^{\ell-1}})$ are both equal to $1+o(h_\ell{}^a)$ for all $a>0$.
Consequently, we deduce that

$$\mathbb{E}[(\overline{Q}_\ell^f - \overline{Q}_\ell^c)^2 \mathbf{1}_{A_2}] = O(h_\ell{}^{2(1-\eta(1+\nu))}).$$

- **Third event** A_3 corresponds to the rest of the non extreme paths.

➜ We prove

$$\left|\prod_{i=0}^{2^\ell-1}(1-\overline{q}_i^{2^\ell})-\prod_{i=0}^{2^\ell-1}(1-\overline{q}_i^{2^{\ell-1}})\right| = O({h_\ell}^{\frac{1}{2}-2\eta(1+\varepsilon)}).$$

Therefore, as we work on the non-extreme paths events, we deduce that

$$|\overline{Q}_\ell^f-\overline{Q}_\ell^c|^2 = \left|g(\overline{Y}_T^{2^\ell})\prod_{i=0}^{2^\ell-1}(1-\overline{q}_i^{2^\ell})-g(\overline{Y}_T^{2^{\ell-1}})\prod_{i=0}^{2^\ell-1}(1-\overline{q}_i^{2^{\ell-1}})\right|^2$$
$$\leq C{h_\ell}^{1-6\eta(1+\varepsilon)-2\eta\nu},$$

$$\mathbb{E}[(\overline{Q}_\ell^f-\overline{Q}_\ell^c)^2\mathbf{1}_{A_3}] = O({h_\ell}^{1-6\eta(1+\varepsilon)-2\eta\nu}\times\mathbb{P}(|\inf_{t\in[0,T]}Y_t-\mathcal{D}|\leq {h_\ell}^{\frac{1}{2}-\eta(1+\varepsilon)}))$$
$$= O({h_\ell}^{\frac{3}{2}-7\eta(1+\varepsilon)-2\eta\nu}).$$

➜ we choose $\varepsilon=\delta$ and $\eta=\frac{\frac{1}{2}-\delta}{7(1+\delta)+2\nu}$, which yields

$$\mathbb{E}[(\overline{Q}_\ell^f-\overline{Q}_\ell^c)^2\mathbf{1}_{A_3}] = O(h^{1+\delta}) \text{ and } \mathbb{E}[(\overline{Q}_\ell^f-\overline{Q}_\ell^c)^2\mathbf{1}_{A_2}] = O(h^{1+\delta}).$$

- Finally, for the first event, we choose $q=2(1+\delta)$ to guarantee that $\mathbb{E}[(\overline{Q}_\ell^f-\overline{Q}_\ell^c)^2\mathbf{1}_{A_1}] = O(h^{1+\delta})$ which is satisfied as soon as

$$p > \frac{14(1+\delta)^2+4(1+\delta)\nu}{\frac{1}{2}-\delta}.$$

Application to the CIR model

- we consider the problem of pricing D-O and U-O barrier options

$$\pi_{\mathcal{D}} = \mathbb{E}\Big[f(X_T)\mathbf{1}_{\{\inf_{t\in[0,T]} X_t > \mathcal{D}\}}\Big] \text{ and } \pi_{\mathcal{U}} = \mathbb{E}\Big[f(X_T)\mathbf{1}_{\{\sup_{t\in[0,T]} X_t < \mathcal{U}\}}\Big],$$

where f is a Lipschitz payoff function with Lipschitz constant $[f]_{\mathrm{Lip}}$ and $(X_t)_{0\leq t\leq T}$ denotes the Cox-Ingersoll-Ross (CIR) process solution to

$$dX_t = (a - \kappa X_t)dt + \sigma\sqrt{X_t}dW_t, \quad X_0 = x > 0 \tag{14}$$

with $a \geq \sigma^2/2$, $\kappa \in \mathbb{R}$, $\sigma > 0$, $X_0 = x > 0$.

- Applying the Lamperti transformation, the process $(Y_t)_{0\leq t\leq T}$ given by $Y_t = \sqrt{X_t}$ satisfies

$$dY_t = L(Y_t)dt + \gamma dW_t, \quad Y_0 = \sqrt{x}, \tag{15}$$

where $L(y) = \dfrac{a - \sigma^2/4}{2y} - \dfrac{\kappa}{2}y$ and $\gamma = \dfrac{\sigma}{2}$.

- Thus, for $g : x \in \mathbb{R} \mapsto g(x) = f(x^2)$ we get

$$\pi_{\mathcal{D}} = \mathbb{E}\Big[g(Y_T)\mathbf{1}_{\{\inf_{t\in[0,T]} Y_t > \sqrt{\mathcal{D}}\}}\Big] \text{ and } \pi_{\mathcal{U}} = \mathbb{E}\Big[g(Y_T)\mathbf{1}_{\{\sup_{t\in[0,T]} Y_t < \sqrt{\mathcal{U}}\}}\Big].$$

- As $a - \sigma^2/4 > 0$, we easily check assumptions (H1) and (H4).
- Besides, noticing that $\lim_{y \to 0^+} L'(y) = \lim_{y \to 0^+} -\frac{(a-\sigma^2/4)}{2y^2} - \frac{\kappa}{2} = -\infty$, we deduce that L is decreasing on $(0, \epsilon)$ for ϵ small enough. It is also globally Lipschitz on $[\varepsilon, +\infty)$ so that assumption (H3) is satisfied with $A = \varepsilon$ and $L'_A = \frac{|a-\sigma^2/4|}{2\varepsilon^2} + \frac{\kappa}{2}$.
- To check $(\tilde{\mathrm{H}}2)$ it is enough to show that

$$\sup_{t \in [0,T]} \mathbb{E}\left[|L'(Y_t)L(Y_t)|^p + |L''(Y_t)|^p + |L'(Y_t)|^{(2 \vee p)} + |L(Y_t)|^p\right] < \infty$$

which is clearly satisfied as soon as

$$\sup_{t \in [0,T]} \mathbb{E}\left[Y_t^{-(4 \vee 3p)}\right] = \sup_{t \in [0,T]} \mathbb{E}\left[X_t^{-(2 \vee \frac{3}{2}p)}\right] < \infty.$$

Recalling that $\sup_{t \in [0,T]} \mathbb{E}\left[X_t^q\right] < \infty$ for all $q > -\frac{2a}{\sigma^2}$ we easily conclude that this holds when $\sigma^2 < a$ and $p < \frac{4}{3}\frac{a}{\sigma^2}$.
- Consequently, for $\delta \in (0, 1/2)$, if $\frac{4}{3}\frac{a}{\sigma^2} > p > \frac{14(1+\delta)^2+4(1+\delta)}{1-2\delta} > 18$ then Theorem 7 is valid provided that $\inf_{t \in [0,T]} Y_t$ (resp. $\sup_{t \in [0,T]} Y_t$) has a bounded density in the neighborhood of the barrier.

- Let us introduce firstly the confluent hypergeometric function $_1F_1(x, b, y)$ defined for all y, $x \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$ by

$$_1F_1(x, b, y) = \sum_{n=0}^{\infty} \frac{(x)_n}{(b)_n n!} y^n,$$

where $(x)_n = x(x+1)...(x+n-1)$ stands for the Pochhammer symbol.

Theorem 5

Let $(X_t)_{0 \leq t \leq T}$ denote the CIR process solution to (14). Then $\sup_{t\in[0,T]} X_t$ has a continuous density on any compact set $K \subset (X_0, +\infty)$, given by

$$z \in K \mapsto P_{\mathrm{CIR,Max}}(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(1+iu)T} \hat{\phi}(u, z) du$$

with

$$\hat{\phi}(u, z) = \frac{{}_1F_1((1+iu)/\kappa, 2a/\sigma^2, 2\kappa X_0/\sigma^2)\,{}_1F_1((1+iu)/\kappa + 1, 2a/\sigma^2 + 1, 2\kappa z/\sigma^2)}{a\,{}_1F_1((1+iu)/\kappa, 2a/\sigma^2, 2\kappa z/\sigma^2)^2}.$$

- To do so, we introduce the Tricomi confluent hypergeometric function $U(a,b,z)$ defined for all a, $z \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \{\pm 0, \pm 1, \pm 2, \dots\}$ by

$$U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)} {}_1F_1(a,b,z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} {}_1F_1(1+a-b, 2-b, z).$$

- Let us denote by $\tau_{X_0 \downarrow z} := \inf\{t \geq 0 : X_t = z\}$ for $0 < z < X_0$. By

[Chou and Lin 2006]

$$\mathbf{E}[e^{-s\tau_{X_0\downarrow z}}] = \frac{U(s/\kappa, 2a/\sigma^2, 2\kappa X_0/\sigma^2)}{U(s/\kappa, 2a/\sigma^2, 2\kappa z/\sigma^2)}, \text{ for } s > 0.$$

Theorem 6

The running minimum $\inf_{t\in[0,T]} X_t$ *has a continuous density on any compact set* $K \subset (0, X_0)$, *given by*

$$z \in K \mapsto P_{\mathrm{CIR,Min}}(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(1+iu)T} \hat{\psi}(u,z) du$$

with

$$\hat{\psi}(u,z) = \frac{2U((1+iu)/\kappa, 2a/\sigma^2, 2\kappa X_0/\sigma^2) U((1+iu)/\kappa + 1, 2a/\sigma^2 + 1, 2\kappa z/\sigma^2)}{\sigma^2 U((1+iu)/\kappa, 2a/\sigma^2, 2\kappa z/\sigma^2)^2}.$$

Numerical Tests

- we consider the problem of pricing D-O and U-O barrier options $\pi_{\mathcal{D}} = \mathbb{E}\Big[f(X_T)\mathbf{1}_{\{\inf_{t\in[0,T]} X_t > \mathcal{D}\}}\Big]$ and $\pi_{\mathcal{U}} = \mathbb{E}\Big[f(X_T)\mathbf{1}_{\{\sup_{t\in[0,T]} X_t < \mathcal{U}\}}\Big]$, where the payoff function $f(x) = e^{-rT}(x-K)_+$.
- By the Lamperti transform we get

$$\pi_{\mathcal{D}} = \mathbb{E}\Big[g(Y_T)\mathbf{1}_{\{\inf_{t\in[0,T]} Y_t > \sqrt{\mathcal{D}}\}}\Big] \text{ and } \pi_{\mathcal{U}} = \mathbb{E}\Big[g(Y_T)\mathbf{1}_{\{\sup_{t\in[0,T]} Y_t < \sqrt{\mathcal{U}}\}}\Big],$$

where $g(x) = e^{-rT}(x^2-K)_+$ and $(Y_t)_{t\in[0,T]}$.

- We consider our interpolated drift implicit scheme

$$\overline{Y}_t^n = \overline{Y}_{t_i}^n + \left(\frac{a-\gamma^2}{2\overline{Y}_{t_{i+1}}^n} - \frac{\kappa}{2}\overline{Y}_{t_{i+1}}^n\right)(t-t_i) + \gamma(W_t - W_{t_i}), \qquad \text{for } t \in [t_i, t_{i+1}],$$

$$Y_0 = \sqrt{X_0} \text{ and } \gamma = \frac{\sigma}{2}.$$

For n large enough, the positive solution is

$$\overline{Y}_{t_{i+1}}^n = \frac{\sqrt{(2+\kappa\frac{T}{n})(a-\gamma^2)\frac{T}{n} + (\gamma(W_{t_{i+1}} - W_{t_i}) + \overline{Y}_{t_i}^n)^2} + \gamma(W_{t_{i+1}} - W_{t_i}) + \overline{Y}_{t_i}^n}{2+\kappa\frac{T}{n}}.$$

Numerical tests

- We take $r = 0.1$, $X_0 = 100$, $a = 0$, $\kappa = -0.1$, $\sigma = 2.5$ and $T = 0.5$. For the D-O option the strike is $K = 95$, and the barrier $\mathcal{D} = 90$ and for the U-O option the strike is $K = 105$ and the barrier $\mathcal{U} = 120$.

- The benchmark prices given in [Davydov and Linetsky 2001] for the D-O (resp. U-O) option is 10.6013 (resp. 0.7734).

- The performance of the improved MLMC is given in the tables and figure below.

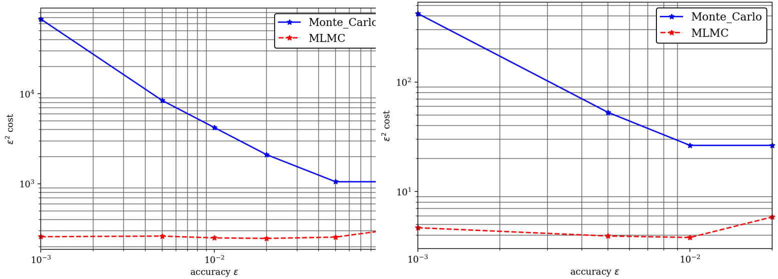
Accuracy	Price	MLMC cost	MC cost	Saving
10^{-3}	10.669	2.588×10^{8}	6.752×10^{10}	260.91
5×10^{-3}	10.668	1.051×10^{7}	3.376×10^{8}	32.13
10^{-2}	10.668	2.510×10^{6}	4.220×10^{7}	16.81
2×10^{-2}	10.677	6.187×10^{5}	5.275×10^{6}	8.52

Table: MLMC complexity tests for D-O barrier option pricing of $\pi_{\mathcal{D}}$

Numerical tests

Accuracy	Price	MLMC cost	MC cost	Saving
10^{-3}	0.77200	4.674×10^{6}	4.221×10^{8}	90.32
5×10^{-3}	0.76926	1.571×10^{5}	2.11×10^{6}	13.44
10^{-2}	0.77015	3.809×10^{4}	2.638×10^{5}	6.93
2×10^{-2}	0.78168	1.463×10^{4}	6.596×10^{4}	4.51

Table: MLMC complexity tests for U-O barrier option pricing $\pi_{\mathcal{U}}$



(a) Approximation of $\pi_{\mathcal{D}}$ (b) Approximation of $\pi_{\mathcal{U}}$

Figure: Comparison for the performances of MLMC vs classical MC algorithm

Application to the CEV model

- For CEV process solution to

$$dX_t = \mu X_t dt + \sigma X_t^{\alpha} dW_t, \quad t \geq 0, \quad , X_0 > 0, \quad \mu \in \mathbb{R} \quad \text{and } \alpha > 1$$

we consider the problem of pricing an U-O barrier option

$$\Pi_{\mathcal{D}}^{\text{U-O},X} := \mathbb{E}\Big[f(X_T)\mathbf{1}_{\{\sup_{t\in[0,T]} X_t < \mathcal{D}\}}\Big] \text{ and } \Pi_{\mathcal{U}}^{\text{D-O},X} := \mathbb{E}\Big[f(X_T)\mathbf{1}_{\{\inf_{t\in[0,T]} X_t > \mathcal{U}\}}\Big]$$

f is a given Lipschitz function with Lipschitz constant $[f]_{\text{Lip}}$.

- For $\alpha > 1$, by Feller's test the solution $(X_t)_{t\in[0,T]}$ is positive.

- So applying the Lamperti transformation, $Y_t = X_t^{1-\alpha}$ is well defined on $I = (0, +\infty)$ and satisfies

$$dY_t = L(Y_t)dt + \gamma dW_t, \quad Y_0 = X_0^{1-\alpha}$$

where $L(y) = (1-\alpha)\left(\mu y - \alpha \frac{\sigma^2}{2} y^{-1}\right)$ and $\gamma = \sigma(1-\alpha)$ and thus

$$\Pi_{\mathcal{D}}^{\text{U-O},X} = \mathbb{E}\Big[g(Y_T)\mathbf{1}_{\{\inf_{t\in[0,T]} Y_t > \mathcal{D}^{1-\alpha}\}}\Big] \text{ and } \Pi_{\mathcal{U}}^{\text{D-O},X} = \mathbb{E}\Big[g(Y_T)\mathbf{1}_{\{\sup_{t\in[0,T]} Y_t < \mathcal{U}^{1-\alpha}\}}\Big],$$

with $g : x \in \mathbb{R} \mapsto f(x^{\frac{1}{1-\alpha}})$.

- As $\lim_{y\to 0^+} L'(y) = \lim_{y\to 0^+} (1-\alpha)(\mu + \alpha\frac{\sigma^2}{2}y^{-2}) = -\infty$, we deduce that L is decreasing on $(0,\epsilon)$ for $\epsilon > 0$ small enough and it is clearly globally Lipschitz on $[\varepsilon, +\infty)$ so that assumption (H3) is satisfied.

- On the one hand, by Itô's formula the process $(Z_t)_{0\le t\le T}$ given by $Z_t = \frac{X_t^{-2(\alpha-1)}}{4(\alpha-1)^2}$ is a CIR process solution to

$$dZ_t = (a - \kappa Z_t)dt - \sigma\sqrt{Z_t}dW_t, Z_0 = \frac{X_0^{-2(\alpha-1)}}{4(\alpha-1)^2},$$

with $a = \frac{\sigma^2(2\alpha-1)}{4(\alpha-1)}$ and $\kappa = 2\mu(\alpha-1)$. Thanks to this second transformation we deduce that $\sup_{t\in[0,T]} \mathbb{E}[Y_t^q] < \infty$ for $q > -\frac{2\alpha-1}{2(\alpha-1)}$.

- On the other hand to check assumption $(\tilde{\mathrm{H}}2)$ it is enough to show that

$$\sup_{t\in[0,T]} \mathbb{E}\left[|L'(Y_t)L(Y_t)|^p + |L''(Y_t)|^p + |L'(Y_t)|^{(2\vee p)} + |L(Y_t)|^p\right] < \infty$$

which is satisfied if $\sup_{t\in[0,T]} \mathbb{E}\left[Y_t^{-(4\vee 3p)}\right] < \infty$. This condition is satisfied when $4 < \frac{2\alpha-1}{2(\alpha-1)}$ (i.e. $\alpha \in (1, \frac{7}{6})$) and $p < \frac{2\alpha-1}{6(\alpha-1)}$.

Corollary 7

- *For $\alpha > 1$, let $(Y_t)_{t\geq 0}$ denotes the Lamperti transform of the CEV process $(X_t)_{t\geq 0}$ solution to*

$$dY_t = L(Y_t)dt + \gamma dW_t, \quad Y_0 = X_0^{1-\alpha}$$

where $L(y) = (1-\alpha)\left(\mu y - \alpha\frac{\sigma^2}{2} y^{-1}\right)$ and $\gamma = \sigma(1-\alpha)$.

- *Let $g : x \in \mathbb{R} \mapsto f(x^{\frac{1}{1-\alpha}})$ denotes the payoff function with f a continuous Lipschitz function. Moreover, for $\delta \in (0, 1/2)$, let us choose α close enough to 1 s.t.*

$$\frac{2\alpha - 1}{6(\alpha - 1)} > \frac{14(1+\delta)^2}{\frac{1}{2} - \delta} > 28.$$

If in addition $\inf_{t\in[0,T]} Y_t$ has a bounded density in the neighborhood of the barrier $\mathcal{D}$, then the MLMC estimator $\bar{Q}_{\mathcal{D}}$ given by (12) for the D-O barrier option satisfies

$$\mathrm{Var}(\overline{\mathrm{Q}}_\ell^{\mathrm{f}} - \overline{\mathrm{Q}}_\ell^{\mathrm{c}}) = \mathrm{O}(\mathrm{h}_\ell^{1+\delta}).$$

Remark.

- One can also consider the CEV process for $\alpha \in (\frac{1}{2}, 1)$ solution to

$$dX_t = (a - \kappa X_t)dt + \sigma Y_t^{\alpha} dW_t, \text{ with } X_0 > 0, a > 0.$$

It can be easily checked that for $a > 0$ this SDE is well defined on $I = (0, +\infty)$.

- However, all the conditions of Theorem 4 apply except the condition that $\inf_{t\in[0,T]} X_t$ or $\sup_{t\in[0,T]} X_t$ admits a continuous density in the neighborhood of the barrier seems to be a challenging problem.

Running maximum of the CEV process

- Let us denote by $\tau_{X_0 \uparrow z} := \inf\{t \geq 0 : X_t = z\}$ the first time that the CEV process $(X_t)_{t \geq 0}$ starting at X_0 hits the level $z > X_0$.

- From [Jeanblanc, Yor and Chesney 2009], the Laplace transform of the hitting time $\tau_{X_0 \uparrow z}$ is given by

$$\mathbf{E}[e^{-s\tau_{X_0 \uparrow z}}] = \left(\frac{X_0}{z}\right)^{\beta + \frac{1}{2}} \exp\left(\frac{\epsilon}{2} c(X_0^{-2\beta} - z^{-2\beta})\right) \frac{W_{k,n}(cX_0^{-2\beta})}{W_{k,n}(cz^{-2\beta})},$$

with $\epsilon = \mathrm{sign}(\mu\beta)$, $n = \frac{1}{4\beta}$, $k = \epsilon\left(\frac{1}{2} + \frac{1}{4\beta}\right) - \frac{s}{2|\mu\beta|}$ and $W_{k,n}$ the Whittaker's function $W_{k,n}(y) = y^{n+\frac{1}{2}} e^{-y/2} U(n - k + \frac{1}{2}, 2n + 1, y)$, where U denotes the confluent hypergeometric function of second kind and $\beta = \alpha - 1$ and $c = \dfrac{|\mu|}{\beta\sigma^2}$.

Theorem 8

Let $(X_t)_{0\leq t\leq T}$ denotes the CEV process solution to (26). Then $\sup_{t\in[0,T]} X_t$ has a continuous density on any compact set $K \subset (X_0, +\infty)$, given by

$$z \in K \mapsto P_{CEV,\ Max}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(1+iu)T}\hat{\Phi}(z,u)du,$$

with

$$\hat{\Phi}(z,u) = -\frac{c}{\mu} z^{-2\beta-1} \frac{U(\frac{1+iu}{2\mu\beta}, 1+\frac{1}{2\beta}, cX_0^{-2\beta})U(\frac{1+iu}{2\mu\beta}+1, 2+\frac{1}{2\beta}, cz^{-2\beta})}{U(\frac{1+iu}{2\mu\beta}, 1+\frac{1}{2\beta}, cz^{-2\beta})^2}, \ \textit{for } \mu > 0$$

and

$$\hat{\Phi}(z,u) = -cz^{-2\beta-1}\left(\frac{2\beta+1}{1+iu} - \frac{1}{\mu}\right) \times \frac{U(1+\frac{1}{2\beta}-\frac{1+iu}{2\mu\beta}, 1+\frac{1}{2\beta}, cX_0^{-2\beta})U(2+\frac{1}{2\beta}-\frac{1+iu}{2\mu\beta}, 2+\frac{1}{2\beta}, cz^{-2\beta})}{U(1+\frac{1}{2\beta}-\frac{1+iu}{2\mu\beta}, 1+\frac{1}{2\beta}, cz^{-2\beta})^2},$$

for $\mu < 0$.

Running minimum of the CEV process

- Let us denote by $\tau_{X_0\downarrow z} := \inf\{t \geq 0 : X_t = z\}$ the first time that the CEV process $(X_t)_{t\geq 0}$ starting at X_0 hits the level $0 < z < X_0$.

- By [Jeanblanc, Yor and Chesney 2009] the Laplace transform of the hitting time $\tau_{X_0\downarrow z} := \inf\{t \geq 0 : X_t = z\}$ is given by

$$\mathbf{E}[e^{-s\tau_{X_0\downarrow z}}] = \left(\frac{X_0}{z}\right)^{\beta+\frac{1}{2}} \exp\left(\frac{\epsilon}{2}c(X_0^{-2\beta} - z^{-2\beta})\right)\frac{M_{k,n}(cX_0^{-2\beta})}{M_{k,n}(cz^{-2\beta})}$$

with $\epsilon = \mathrm{sign}(\mu\beta)$, $n = \frac{1}{4\beta}$, $k = \epsilon\left(\frac{1}{2} + \frac{1}{4\beta}\right) - \frac{s}{2\beta|\mu|}$ and the Whittaker function

$$M_{k,n}(y) = y^{n+\frac{1}{2}}e^{-\frac{y}{2}}{}_1F_1(n - k + \frac{1}{2}, 2n + 1, y),$$

where ${}_1F_1$ denotes the confluent hypergeometric function of the first kind with $\beta = \alpha - 1$ and $c = \dfrac{|\mu|}{\beta\sigma^2}$.

Running minimum of the CEV process

Theorem 9

Let $(X_t)_{0\leq t\leq T}$ denotes the CEV process solution to (26). Then $\inf_{t\in[0,T]} X_t$ has a continuous density on any compact set $K \subset (0, X_0)$, given by

$$z \in K \mapsto P_{CEV,\ Min}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(1+iu)T}\hat{\Psi}(z,u)du,$$

with

$$\hat{\Psi}(z,u) = \frac{cz^{-2\beta-1}}{\mu(1+\frac{1}{2\beta})} \frac{{}_1F_1(\frac{1+iu}{2\mu\beta}, 1+\frac{1}{2\beta}, cX_0^{-2\beta})\,{}_1F_1(\frac{1+iu}{2\mu\beta}+1, 2+\frac{1}{2\beta}, cz^{-2\beta})}{{}_1F_1(\frac{1+iu}{2\mu\beta}, 1+\frac{1}{2\beta}, cz^{-2\beta})^2}, \text{ for } \mu > 0$$

and

$$\hat{\Psi}(z,u) = cz^{-2\beta-1}\left(\frac{2\beta}{1+iu} - \frac{1}{\mu(1+\frac{1}{2\beta})}\right) \times \frac{{}_1F_1(1+\frac{1}{2\beta}-\frac{1+iu}{2\mu\beta}, 1+\frac{1}{2\beta}, cX_0^{-2\beta})\,{}_1F_1(2+\frac{1}{2\beta}-\frac{1+iu}{2\mu\beta}, 2+\frac{1}{2\beta}, cz^{-2\beta})}{{}_1F_1(1+\frac{1}{2\beta}-\frac{1+iu}{2\mu\beta}, 1+\frac{1}{2\beta}, cz^{-2\beta})^2}, \text{ for } \mu < 0.$$

- we used our interpolated drift implicit scheme

$$\overline{Y}^n_t = \overline{Y}^n_{t_i} + (1-\alpha)\left(\mu\overline{Y}^n_{t_{i+1}} - \alpha\frac{\sigma^2}{2\overline{Y}^n_{t_{i+1}}}\right)(t-t_i) + \gamma(W_t - W_{t_i}), \text{ for } t \in [t_i, t_{i+1}[, 0 \leq i \leq n$$

$Y_0 = {X_0}^{1-\alpha}$, and $\gamma = \sigma(1-\alpha)$.

- For n large enough, the positive solution to the above implicit scheme is explicit and given by

$$\overline{Y}^n_{t_{i+1}} = \frac{\sqrt{2\sigma^2\alpha(\alpha-1)(1+\mu(\alpha-1)\frac{T}{n})\frac{T}{n} + (\gamma(W_{t_{i+1}} - W_{t_i}) + \overline{Y}^n_{t_i})^2} + \gamma(W_{t_{i+1}} - W_{t_i}) + \overline{Y}^n_{t_i}}{2 + 2\mu(\alpha-1)\frac{T}{n}}.$$

- We choose $\alpha = 1.2$, $X_0 = 100$, $\mu = 0.1$, $\sigma = 0.2$, $T = 1$. The payoff

function $g(x) = e^{-rT}(x^{\frac{1}{1-\alpha}} - K)_+$ is a discounted call function with $r = 0.1$. For the U-O option the strike is $K = 90$, and the barrier $\mathcal{D} = 150$. For the D-O option the strike is $K = 100$ and the barrier $\mathcal{U} = 90$.

Numerical tests

The tables and the figures below confirm the high performance of the improved MLMC.

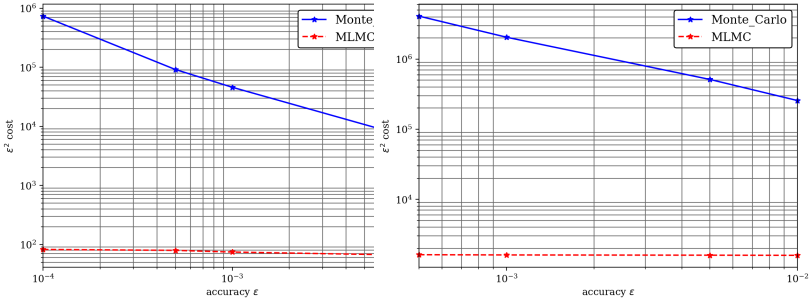
Accuracy	Price	MLMC cost	MC cost	Saving
10^{-4}	3.0390	8.226×10^9	7.34×10^{13}	8922.33
5×10^{-4}	3.0391	3.17×10^8	3.67×10^{11}	1155.67
10^{-3}	3.041	7.436×10^7	4.587×10^{10}	616.91
10^{-2}	3.0452	6.539×10^5	5.734×10^7	87.69

Table: MLMC complexity tests for the U-O barrier option pricing of $\Pi_{\mathcal{D}}^{\text{U-O},X}$

Accuracy	Price	MLMC cost	MC cost	Saving
5×10^{-4}	11.102	6.483×10^9	1.642×10^{13}	2532.83
10^{-3}	11.103	1.608×10^9	2.053×10^{12}	1276.66
5×10^{-3}	11.106	6.379×10^7	2.053×10^{10}	321.77
10^{-2}	11.094	1.587×10^7	2.566×10^9	161.69

Table: MLMC complexity tests for the D-O barrier option pricing of $\Pi_{\mathcal{U}}^{\text{D-O},X}$

Numerical tests



(a) Approximation of $\Pi_{\mathcal{D}}^{\text{U-O},X}$ (b) Approximation of $\Pi_{\mathcal{U}}^{\text{D-O},X}$

Figure: Comparison for the performances of MLMC vs classical MC algorithm under the CEV model

Reference

- Ben Derouich, M. and Kebaier, A: The interpolated drift implicit Euler scheme Multilevel Monte Carlo method for pricing Barrier options and applications to the CIR and CEV models https://arxiv.org/abs/2210.00779 (2022).

Thank you !