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AGH University of Krakow

On error bounds, optimality and exceptional sets for selected randomized schemes for ODEs

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Stochastic Computation and Complexity at MCM 2023
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Agenda

0 Introduction

1 Error bounds under inexact information

2 Exceptional set

3 Optimality

Initial Value Problem

$$\begin{cases} z'(t) = f(t, z(t)), & t \in [a, b], \\ z(a) = \eta, \end{cases}$$

- $-\infty < a < b < \infty$
- $\eta \in \mathbb{R}^d$
- $d \in \mathbb{N}$
- $f : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d$

Randomized Euler schemes

$n \in \mathbb{Z}_+,\ h = \frac{b-a}{n},\ t_j = a + jh$ for $j \in \{0, 1, \ldots, n\}$

$\tau_1, \tau_2, \ldots : \Omega \to \mathbb{R}$ – i.i.d. $U([0,1])$, $\quad \theta_j = t_{j-1} + \tau_j h$

Randomized explicit Euler scheme

$$\begin{cases} U^0 = \eta, \\ U^j = U^{j-1} + h \cdot f\left(\theta_j, U^{j-1}\right),\ j \in \{1, \ldots, n\} \end{cases}$$

$$I^{EE}(t) = \frac{U^j - U^{j-1}}{h}(t - t_{j-1}) + U^{j-1} \text{ for } t \in [t_{j-1}, t_j], j \in \{1, \ldots, n\}$$

Randomized implicit Euler scheme

$$\begin{cases} W^0 = \eta, \\ W^j = W^{j-1} + h \cdot f\left(\theta_j, W^j\right),\ j \in \{1, \ldots, n\} \end{cases}$$

$$I^{IE}(t) = \frac{W^j - W^{j-1}}{h}(t - t_{j-1}) + W^{j-1} \text{ for } t \in [t_{j-1}, t_j], j \in \{1, \ldots, n\}$$

Randomized RK2 scheme

$n \in \mathbb{Z}_+,\ h = \frac{b-a}{n},\ t_j = a + jh$ for $j \in \{0, 1, \ldots, n\}$

$\tau_1, \tau_2, \ldots : \Omega \to \mathbb{R}$ – i.i.d. $U([0,1])$, $\quad \theta_j = t_{j-1} + \tau_j h$

Randomized two-stage Runge-Kutta (midpoint) scheme

$$\begin{cases} V^0 := \eta, \\ V^j_\tau := V^{j-1} + \tau_j h f(t_{j-1}, V^{j-1}),\ j \in \{1, \ldots, n\}, \\ V^j := V^{j-1} + h f(\theta_j, V^j_\tau),\ j \in \{1, \ldots, n\} \end{cases}$$

$$I^{RK}(t) = \frac{V^j - V^{j-1}}{h}(t - t_{j-1}) + V^{j-1} \text{ for } t \in [t_{j-1}, t_j], j \in \{1, \ldots, n\}$$

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Random Error: $\sup_{t\in[a,b]} \|\mathcal{A}(t) - z(t)\|$

$L^p(\Omega)$ error: $\|\text{Random Error}\|_{L^p(\Omega)} = \left(\mathbb{E}(\text{Random Error})^p\right)^{1/p}$

Assumptions about the right-hand side function

Let $K, L \in (0, \infty)$, $\varrho \in (0, 1]$.

Consider a class F_R^{ϱ} of pairs (η, f) such that:

(A0) $\|\eta\| \leqslant K$,

(A1) $f \in C\left([a, b] \times \mathbb{R}^d\right)$,

(A2) $\|f(t, x)\| \leqslant K(1 + \|x\|)$, for all $(t, x) \in [a, b] \times \mathbb{R}^d$,

(A3) $\|f(t, x) - f(s, x)\| \leqslant L|t - s|^{\varrho}$ for all $t, s \in [a, b]$, $x \in \bar{B}(\eta, R)$,

(A4) $\|f(t, x) - f(t, y)\| \leqslant L\|x - y\|$ for all $t \in [a, b]$, $x, y \in \bar{B}(\eta, R)$.

Parameters of the class F_R^{ϱ}: $a, b, d, \varrho, K, L, R$.

Error bound for the rand. EE scheme

Assume that:

- $(\eta, f) \in F^{\varrho}_{R_{EE}}$, $R_{EE} = \max\left\{(K+2)e^{(K+1)(b-a)} + K - 1, K(1+b-a)e^{K(b-a)} + K\right\}$.
- We have access only to noisy evaluations of f: $\tilde{f}(t,y) = f(t,y) + \tilde{\delta}(t,y)$, where $\|\tilde{\delta}(t,y)\| \leqslant \delta \cdot (1 + \|y\|)$ for all $t \in [a,b]$, $y \in \mathbb{R}^d$, $\delta \in [0,1]$.

Theorem

For every $p \in [2, \infty)$,
there exists a constant $C = C(a,b,d,K,L,\varrho,p) > 0$
such that for all $n \geqslant \lfloor b-a \rfloor + 1$, all $\delta \in [0,1]$,
and all IVPs and noise functions satisfying above assumptions:

$$\left\| \sup_{a \leqslant t \leqslant b} \left\| z(\eta,f)(t) - I^{EE}(\tilde{\eta},\tilde{f})(t) \right\| \right\|_{L^p(\Omega)} \leqslant C\left(h^{\min\left\{\varrho + \frac{1}{2}, 1\right\}} + \delta \right).$$

R. Kruse, Y. Wu. *Error analysis of randomized Runge-Kutta methods for differential equations with time-irregular coefficients*. Comput. Methods Appl. Math., 17, 479–498, 2017.

TB, M. Goćwin, P. M. Morkisz, P. Przybyłowicz. *Randomized Runge-Kutta method – Stability and convergence under inexact information*. J. Complex. 65, 101554, 2021.

Error bound for the rand. IE scheme

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Assume that:

- $(\eta, f) \in F_{\infty}^{\varrho}$.
- We have access only to noisy evaluations of f: $\tilde{f}(t,y) = f(t,y) + \tilde{\delta}(t,y)$, where $\|\tilde{\delta}(t,y)\| \leqslant \delta \cdot (1 + \|y\|)$ and $\|\tilde{\delta}(t,y_1) - \tilde{\delta}(t,y_2)\| \leqslant \delta \|y_1 - y_2\|$ for all $t \in [a,b]$, $y, y_1, y_2 \in \mathbb{R}^d$, $\delta \in [0,1]$.

Theorem

For every $p \in [2, \infty)$,
there exists a constant $C = C(a,b,d,K,L,\varrho,p) > 0$
such that for all n with $h(K+1) \leqslant \frac{1}{2}$ and $hL \leqslant \frac{1}{2}$, all $\delta \in [0,1]$,
and all IVPs and noise functions satisfying above assumptions:

$$\left\| \sup_{a \leqslant t \leqslant b} \left\| z(\eta, f)(t) - I^{IE}(\tilde{\eta}, \tilde{f})(t) \right\| \right\|_{L^p(\Omega)} \leqslant C \left(h^{\min\{\varrho + \frac{1}{2}, 1\}} + \delta \right).$$

TB, P. Przybyłowicz. *On the randomized Euler schemes for ODEs under inexact information.* Numer. Algorithms, 91, 1205–1229, 2022.

Error bound for the rand. RK2 scheme

Assume that:

- $(\eta, f) \in F^{\varrho}_{R_{RK}}$,
  $R_{RK} = \max\Big\{R_1 + K(b-a)(1 + R_1 - K),\ (K_0+1)\Big((K+2)e^{K_0(K_0+1)} - 1\Big) + K_0 + K\Big\}.$
- We have access only to noisy evaluations of f:
  $\tilde{f}(t,y) = f(t,y) + \tilde{\delta}(t,y)$, where $\|\tilde{\delta}(t,y)\| \leqslant \delta \cdot (1 + \|y\|)$ for all $t \in [a,b]$, $y \in \mathbb{R}^d$, $\delta \in [0,1]$.

Theorem

For every $p \in [2,\infty)$, there is a constant $C = C(a,b,d,K,L,\varrho,p) > 0$ such that for all n, all $\delta \in [0,1]$,
and all IVPs and noise functions satisfying above assumptions:

$$\Big\| \sup_{a \leqslant t \leqslant b} \big\| z(\eta,f)(t) - I^{RK}(\tilde{\eta},\tilde{f})(t) \big\| \Big\|_{L^p(\Omega)} \leqslant C\Big(h^{\varrho + \frac{1}{2}} + \delta\Big).$$

S. Heinrich, B. Milla, *The randomized complexity of initial value problems*. J. Complex. 24, 77–88, 2008.

R. Kruse, Y. Wu. *Error analysis of randomized Runge-Kutta methods for differential equations with time-irregular coefficients*. Comput. Methods Appl. Math., 17, 479–498, 2017.

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Numerical example: empirical rate of convergence

Test problem: $\begin{cases} z'(t) = |\sin(100t)|^r \cdot z(t) \cdot \sin\big(z(t)^2\big), & t \in [-1, 2], \\ z(-1) = 1. \end{cases}$

Algorithm	explicit Euler			implicit Euler		
Sch./inf.	$r = 0.25$	$r = 0.5$	$r = 0.75$	$r = 0.25$	$r = 0.5$	$r = 0.75$
det./ex.	1.026	1.079	1.073	1.025	1.037	1.008
det./inex.	0.788	1.047	1.047	0.774	1.014	0.985
rand./ex.	1.151	1.286	1.349	1.154	1.283	1.345
rand./inex.	0.815	1.199	1.232	0.811	1.195	1.225

Test problem: $\begin{cases} z'(t) = t^r \cdot z(t) \cdot \sin\big(z(t)^2\big), & t \in [0, 1], \\ z(0) = 1. \end{cases}$

Algorithm	two-stage Runge-Kutta		
Sch./inf.	$r = 0.25$	$r = 0.5$	$r = 0.75$
det./ex.	1.186	1.388	1.534
det./inex.	0.742	0.996	0.994
rand./ex.	1.278	1.453	1.481
rand./inex.	0.758	1.007	1.257

For the case of inexact information:

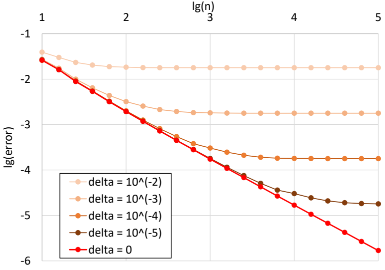
$\delta = h^\gamma$, where $\gamma = \min\{\varrho + 1/2, 1\}$ for EE and IE, $\gamma = \varrho + 1/2$ for RK2.

Numerical example: effect of noisy information

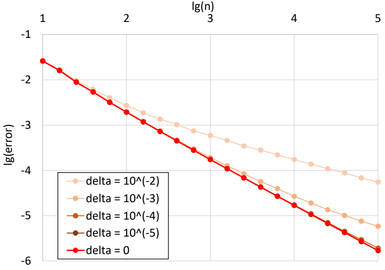
Constant value of the noise parameter δ

- *worst-case noise*: $\tilde{f}(t,y) = f(t,y) \pm \delta(1+|y|)$ depending on which option gives larger error
- *random noise*: $\tilde{f}(t,y) = f(t,y) + D(1+|y|)$, where D is a random number from $[-\delta, \delta]$

Algorithm: rand. explicit Euler



worst-case noise



random noise

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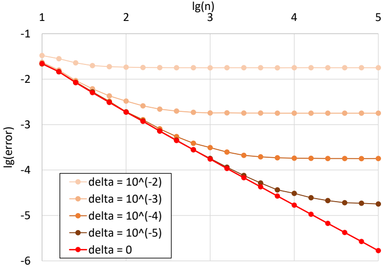
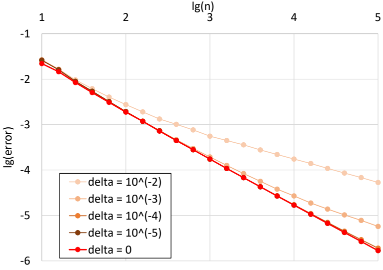


worst-case noise



random noise

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Constant value of the noise parameter δ

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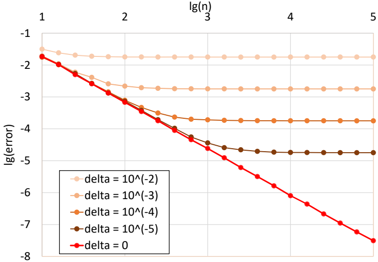
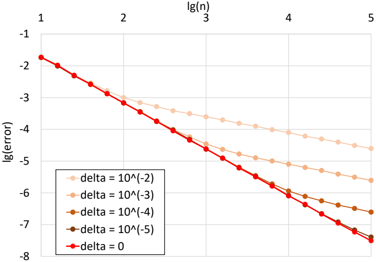
- *random noise*: $\tilde{f}(t,y) = f(t,y) + D(1+|y|)$, where D is a random number from $[-\delta, \delta]$

Algorithm: rand. RK2



worst-case noise



random noise

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Aim

We have established

$$\left(\mathbb{E}\Big(\sup_{a\leqslant t\leqslant b}\|z(t)-\mathcal{A}_n(t)\|\Big)^p\right)^{1/p}\leqslant C\left(h^\gamma+\delta\right),$$

with $\gamma=\min\{\varrho+\frac{1}{2},1\}$ for $\mathcal{A}\in\{EE,IE\}$ and $\gamma=\varrho+\frac{1}{2}$ for $\mathcal{A}=RK2$.

Now we consider

$$\mathbb{P}\Big(\sup_{a\leqslant t\leqslant b}\|z(t)-\mathcal{A}_n(t)\|>\xi\max\{h^\gamma,\delta\}\Big)\leqslant ?$$

Probability of exceptional set for the rand. EE scheme

Assume that:

- $(\eta, f) \in F^{\varrho}_{R_{EE}}$, $R_{EE} = \max\Big\{(K+2)e^{(K+1)(b-a)} + K - 1, K(1+b-a)e^{K(b-a)} + K\Big\}$.
- We have access only to noisy evaluations of f: $\tilde{f}(t,y) = f(t,y) + \tilde{\delta}(t,y)$, where $\tilde{\delta}$ is Borel meas. and has at most linear growth in y (with parameter $\delta \in [0,1]$).

Fact

There exist constants $c_1, c_2 > 0$ dependent only on a, b, d, K, L,

such that for all for all IVPs and all noise functions satisfying the above assumptions,

for all $n \geqslant \lfloor b-a \rfloor + 1$, and for all $\xi \geqslant c_1$,

$$\mathbb{P}\Big(\sup_{a\leqslant t\leqslant b} \big\|z(t) - l^{EE}(t)\big\| > \xi \max\big\{h^{\min\{\varrho+1/2,1\}}, \delta\big\}\Big) \leqslant \exp(-c_2\xi^2).$$

S. Heinrich, B. Milla, *The randomized complexity of initial value problems*. J. Complex. 24, 77–88, 2008.

TB. *On the properties of the exceptional set for the randomized Euler and Runge-Kutta schemes*. Adv. Comput. Math., 49, 14, 2023.

Probability of exceptional set for rand. IE and RK2 schemes

Fact

There exist constants $c_1, c_2 > 0$ dependent only on a, b, d, K, L, such that for all IVPs and all noise functions considered in the error analysis for the rand. IE scheme,
for all n satisfying $h(K+1) \leqslant \frac{1}{2}$ and $hL \leqslant \frac{1}{2}$, and for all $\xi \geqslant c_1$,

$$\mathbb{P}\left(\sup_{a\leqslant t\leqslant b} \left\|z(t) - l^{IE}(t)\right\| > \xi \max\left\{h^{\min\{\varrho+1/2,1\}}, \delta\right\}\right) \leqslant \exp(-c_2\xi^2).$$

Fact

There exist constants $c_1, c_2 > 0$ dependent only on a, b, d, K, L, such that for all IVPs and all noise functions considered in the error analysis for the rand RK2 scheme,
for all $n \geqslant \lfloor b-a \rfloor + 1$, and for all $\xi \geqslant c_1$,

$$\mathbb{P}\left(\sup_{a\leqslant t\leqslant b} \left\|z(t) - l^{RK}(t)\right\| > \xi \max\left\{h^{\varrho+1/2}, \delta\right\}\right) \leqslant \exp(-c_2\xi^2).$$

Numerical example: exceptional set – part 1

- $\mathbb{P}\Big(\sup\limits_{a\leqslant t\leqslant b}\|z(t)-\mathcal{A}_n(t)\| > \xi(\varepsilon)\cdot\max\{n^{-\gamma},\delta\}\Big) = \varepsilon$
- Set ε, n, and δ. Assume that the exact solution z is known.
- Simulate N values $r(n,\delta) = \sup\limits_{a\leqslant t\leqslant b}\|z(t)-\mathcal{A}_n(t)\|$ and sort them increasingly: $r_{1:N} \leqslant \ldots \leqslant r_{N:N}$.
- Consider the following estimator of $\xi(\varepsilon)$:

$$\hat{\xi}(\varepsilon,n,\delta) = r_{\lceil(1-\varepsilon)N\rceil:N}\cdot\Big(\max\{n^{-\gamma},\delta\}\Big)^{-1}$$

- Check whether $\hat{\xi}$ is ‘stable’ for different choices of n and δ.

Numerical example: exceptional set – part 2

Randomized explicit Euler scheme applied to

$$\begin{cases} z'(t) = 2tz(t), \;\; t \in [0,1], \\ z(0) = 1. \end{cases}$$

n	$\delta = 0$	$\delta(n) = n^{-1.1}$	$\delta(n) = n^{-1}$	$\delta(n) = n^{-0.9}$	$\delta = 2 \cdot 10^{-3}$	$\delta = 10^{-4}$
10	2.29	2.82	3.04	2.64	2.30	2.29
20	2.23	2.58	2.79	2.29	2.24	2.23
50	2.12	2.31	2.48	1.86	2.12	2.12
100	2.04	2.17	2.30	1.60	2.06	2.04
200	1.98	2.06	2.17	1.39	2.02	1.98
500	1.92	1.97	2.04	1.18	2.04	1.92
1 000	1.89	1.92	1.97	1.06	1.05	1.89
2 000	1.87	1.89	1.93	0.95	0.56	1.87
5 000	1.85	1.86	1.89	0.84	0.25	1.86

Numerical example: exceptional set – part 2

Randomized explicit Euler scheme applied to

$$\begin{cases} z'(t) = \cos(z^2(t)), \ t \in [0,1], \\ z(0) = 1. \end{cases}$$

n	$\delta = 0$	$\delta(n) = n^{-1.1}$	$\delta(n) = n^{-1}$	$\delta(n) = n^{-0.9}$	$\delta = 2 \cdot 10^{-3}$	$\delta = 10^{-4}$
10	0.166	0.546	0.659	0.636	0.173	0.166
20	0.159	0.411	0.512	0.482	0.170	0.160
50	0.157	0.299	0.376	0.336	0.175	0.157
100	0.156	0.248	0.308	0.258	0.183	0.157
200	0.155	0.216	0.261	0.203	0.195	0.157
500	0.155	0.189	0.221	0.152	0.221	0.158
1 000	0.155	0.177	0.201	0.126	0.125	0.159
2 000	0.155	0.169	0.187	0.106	0.074	0.161
5 000	0.155	0.163	0.175	0.087	0.039	0.164

Numerical example: exceptional set – part 2

Randomized RK2 scheme applied to

$$\begin{cases} z'(t) = 2tz(t), \ \ t \in [0,1], \\ z(0) = 1. \end{cases}$$

n	$\delta = 0$	$\delta(n) = n^{-1.6}$	$\delta(n) = n^{-1.5}$	$\delta(n) = n^{-1.4}$	$\delta = 2 \cdot 10^{-3}$	$\delta = 10^{-4}$
10	4.80	4.95	5.05	4.14	4.79	4.78
20	5.18	5.25	5.29	4.01	5.18	5.19
50	5.45	5.49	5.52	3.79	5.49	5.51
100	5.61	5.63	5.63	3.59	2.86	5.58
200	5.69	5.71	5.72	3.36	1.08	5.71
500	5.75	5.77	5.77	3.09	0.36	5.14
1 000	5.77	5.80	5.79	2.91	0.20	1.84
2 000	5.80	5.81	5.83	2.72	0.13	0.66
5 000	5.86	5.84	5.82	2.50	0.08	0.18

Numerical example: exceptional set – part 2

Randomized RK2 scheme applied to

$$\begin{cases} z'(t) = \cos\left(z^2(t)\right), \;\; t \in [0,1], \\ z(0) = 1. \end{cases}$$

n	$\delta = 0$	$\delta(n) = n^{-1.6}$	$\delta(n) = n^{-1.5}$	$\delta(n) = n^{-1.4}$	$\delta = 2 \cdot 10^{-3}$	$\delta = 10^{-4}$
10	0.485	0.527	0.550	0.461	0.486	0.485
20	0.459	0.480	0.494	0.386	0.462	0.461
50	0.455	0.464	0.471	0.330	0.465	0.458
100	0.460	0.463	0.466	0.301	0.244	0.459
200	0.463	0.464	0.467	0.279	0.102	0.464
500	0.467	0.466	0.469	0.254	0.045	0.419
1 000	0.469	0.469	0.469	0.237	0.030	0.150
2 000	0.471	0.471	0.470	0.221	0.021	0.056
5 000	0.471	0.470	0.472	0.202	0.013	0.018

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Randomized algorithms based on inexact information

- F – some class of IVPs (η, f)
- $\mathcal{K}_\delta$ – some class of noise functions
- Noisy right-hand side function:

$$\tilde{f} = f + \text{noise function}$$

- Vector of inexact information about $(\eta, f) \in F$:

$$N(\tilde{\eta}, \tilde{f}) = [\tilde{\eta}, \tilde{f}(t_1, y_1), \ldots, \tilde{f}(t_i, y_i), \tilde{f}(\theta_1, z_1), \ldots, \tilde{f}(\theta_i, z_i)]$$

- Algorithm based on $N(\tilde{\eta}, \tilde{f})$:

$$\mathcal{A}(\tilde{\eta}, \tilde{f}) = \varphi(N(\tilde{\eta}, \tilde{f})),$$

where $\varphi : \mathbb{R}^{(2i+1)d} \to D([a, b]; \mathbb{R}^d)$

Error of an algorithm

Worst-case error of the algorithm $\mathcal{A}$:

$$e^{(p)}(\mathcal{A}, F, \mathcal{K}_\delta) = \sup_{(\eta,f)} \sup_{(\tilde{\eta},\tilde{f})} \left\| \sup_{a\leqslant t\leqslant b} \|z(\eta, f)(t) - \mathcal{A}(\tilde{\eta}, \tilde{f})(t)\| \right\|_{L^p(\Omega)}$$

n**th minimal error**:

$$e_n^{(p)}(F, \mathcal{K}_\delta) = \inf_{\mathcal{A}} e^{(p)}(\mathcal{A}, F, \mathcal{K}_\delta),$$

where infimum is taken over all algorithms requiring at most n noisy evaluations of f.

An algorithm whose error asymptotically attains the nth minimal error is called an **optimal algorithm**.

Optimality properties for rand. EE, IE, and RK2 schemes

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- "Optimal" = "optimal in the class of IVPs and the class of noise functions assumed in this algorithm's error analysis".
- We have worst-case errors for randomized EE, IE, and RK2 schemes (under some assumptions).
- For all classes of IVPs and noise functions that we considered, the nth minimal error has the lower bound of order $h^{\varrho+1/2} + \delta$.

E. Novak. *Deterministic and Stochastic Error Bounds in Numerical Analysis*. Lecture Notes in Mathematics, vol. 1349, New York, Springer–Verlag, 1988.

TB, P. Przybyłowicz. *On the randomized Euler schemes for ODEs under inexact information*. Numer. Algorithms, 91, 1205–1229, 2022.

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The randomized RK2 scheme is optimal.

Optimality properties for rand. EE, IE, and RK2 schemes

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- We have worst-case errors for randomized EE, IE, and RK2 schemes (under some assumptions).
- For all classes of IVPs and noise functions that we considered, the *n*th minimal error has the lower bound of order $h^{\varrho+1/2} + \delta$.
- Upper bounds that we established for rand. Euler schemes are sharp: the worst-case error of these algorithms has order exactly $h^{\min\{\rho+1/2,1\}} + \delta$. For $\varrho > 1/2$, it is $h + \delta$.

$$\boxed{\text{The randomized IE scheme is optimal} \quad \Leftrightarrow \quad \varrho \leqslant 1/2.}$$

Optimality properties for rand. EE, IE, and RK2 schemes

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- For all classes of IVPs and noise functions that we considered, the *n*th minimal error has the lower bound of order $h^{\varrho+1/2}+\delta$.
- Upper bounds that we established for rand. Euler schemes are sharp: the worst-case error of these algorithms has order exactly $h^{\min\{\varrho+1/2,1\}}+\delta$. For $\varrho>1/2$, it is $h+\delta$.

The randomized EE scheme is optimal if $\varrho\leqslant 1/2$. For $\varrho>1/2$, there is a gap between lower and upper bounds for the *n*th minimal error, although we suppose that the rand. RK2 scheme is optimal also in this (broader) class of IVPs.

Thank you