Infinite-dimensional Wishart processes

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The Wishart distribution

Let

- $n, p \in \mathbb{N}$,
- $\Gamma = (\gamma_{i,j})_{1 \le i \le p, 1 \le j \le n}$ be i.i.d. standard Gaussian random variables,
- $S^{+}(\mathbb{R}^p) = \{A \in \mathbb{R}^{p \times p} \colon A \text{ is positive semi-definite}\}$,
- $S^{++}(\mathbb{R}^p) = \{A \in \mathbb{R}^{p \times p} \colon A \text{ is positive definite}\}$.

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Let X be the $S^+(\mathbb{R}^p)$-valued random variable given by

$$X = \Gamma\Gamma^\mathsf{T},$$

then[1] $X \sim \mathrm{Wishart}_p(n)$ ('X had a Wishart distribution with n degrees of freedom').

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then[1] $X \sim \mathrm{Wishart}_p(n)$ ('X had a Wishart distribution with n degrees of freedom'). Generalizations:

$$X = (\sqrt{Q}\Gamma + A)(\sqrt{Q}\Gamma + A)^T$$

with $Q \in S^+(\mathbb{R}^p)$, $A \in \mathbb{R}^{p \times n}$: $X \sim \mathrm{Wishart}_p(n, Q, A)$.

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- The Laplace transform $S^{+}(\mathbb{R}^p) \ni u \mapsto \mathbb{E}(e^{-\operatorname{tr}(uX)})$ can be given explicitely.
- The Wishart distribution is used in e.g. multidimensional (Bayesian) statistical analysis and random matrix theory.

The Wishart process

Let

- $A \in \mathbb{R}^{p\times p}$,
- $Y_0, Q \in S^{++}(\mathbb{R}^p)$,
- $W\colon [0,\infty)\times\Omega \to \mathbb{R}^{p\times n}$ a standard Brownian motion,

and let $Y\colon [0,\infty)\times\Omega \to \mathbb{R}^{p\times n}$ be the Ornstein-Uhlenbeck process satisfying

$$dY_t = AY_t\,dt + \sqrt{Q}\,dW_t, \quad t \geq 0,$$

i.e., $Y_t = e^{tA}Y_0 + \int_0^t e^{(t-s)A}\sqrt{Q}\,dW_s$.

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i.e., $Y_t = e^{tA}Y_0 + \int_0^t e^{(t-s)A}\sqrt{Q}\,dW_s$. Set

$$X_t = Y_t Y_t^\top, \quad t \geq 0.$$

Then X_t has a Wishart distribution for all $t > 0$. We call $(X_t)_{t\geq 0}$ a Wishart process.

The Wishart process

Theorem (Bru, 1989)

Let $(X_t)_{t\geq 0}$ *be the Wishart process introduced above and* $n \geq p$. *Then there exists an* $\mathbb{R}^{p\times p}$*-standard Brownian motion* $(B_t)_{t\geq 0}$ *such that*

$$dX_t = (nQ + AX_t + X_t A^T)\, dt + \sqrt{Q}\, dB_t X_t + X_t\, dB_t^T \sqrt{Q}, \quad t \geq 0.$$

Proof.

Apply Itô formula to $Y_t Y_t^\mathsf{T}$. Show that martingale part of $dY_t Y_t^\mathsf{T}$ is equal in distribution to $\sqrt{Q}\, dB_t X_t + X_t\, dB_t^\mathsf{T} \sqrt{Q}$. $\square$

The Wishart process

Theorem (Bru, 1991)

If $\alpha \in \mathbb{N} \cup (p-1, \infty)$ *and* $\operatorname{rank}(X_0) \leq \alpha$ *then there exists a unique* $S^+(\mathbb{R}^p)$*-valued (weak) solution to*

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Moreover,

- *the Laplace transform of* X_t, $t \geq 0$, *can be given explicitely and coincides with a Wishart distribution when* $\alpha \in \mathbb{N}$*;*
- *the eigenvalues and -vectors of* X_t, $t \geq 0$, *solve a system of SDEs that can be given explicitely.*

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Theorem (Cuchiero, Filipovic, Mayerhofer, Teichman: 2011)

If a (weak) solution to (Wis_p) *exists for all* $X_0 \in S^{++}(\mathbb{R}^p)$ *then* $\alpha > p - 1$.

Infinite dimensional Wishart processes: why?

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Infinite-dimensional stochastic processes call for infinite-dimensional covariance models.
E.g.: models in financial/energy/commodity markets, i.e., models describing, for every $t_0 \leq t \leq t_1$, the expected cost of buying 1 euro/kWh/kg of silver, given the information at time t_0.

Samples of the forward rate curve f_t for different values of t

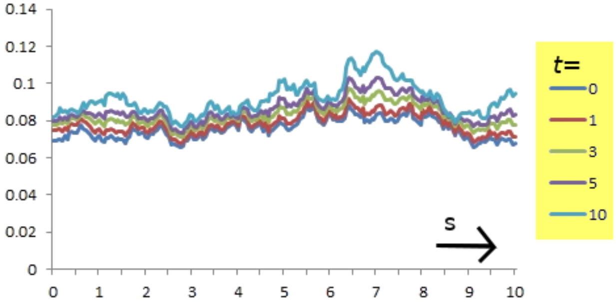

Image source: `http://crazyquant.blogspot.com/2012/10/hjm-model.html`

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Theorem (Bru, 1991)

If $\alpha \in \mathbb{N} \cup (p-1, \infty)$ *and* $\operatorname{rank}(X_0) \leq \alpha$ *then there exists a unique* $S^+(\mathbb{R}^p)$*-valued (stoch. weak) solution to*

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the stochastic covariance is modelled by $X_t := Y_t \otimes Y_t$, where Y is an H-valued Ornstein-Uhlenbeck process:

$$dY_t = AY_t\,dt + \sqrt{Q}dW_t,$$

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- $A\colon D(A) \subseteq H \to H$ is the generator of a C_0-semigroup,
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Note: $X_t \in S_1^+(H)$ a.s.
Indeed, Benth and Simonsen calculate the Laplace transform

$$S_1^+(H) \ni u \mapsto \mathbb{E}e^{-\langle uY_t, Y_t\rangle_H}.$$

This Laplace transform is analogous to that of a Wishart distribution with 1 degree of freedom.

Theorem (C., Cuchiero, Khedher (2023))

Assume

- $A\colon D(A) \subseteq H \to H$ *generator of a* C_0*-semigroup* $(e^{tA})_{t\geq 0}$*,*
- $Q \in S^+(H)$ *such that* $\int_0^t \|Qe^{sA}\|_{L_2(H)}\, ds < \infty$ *for all* $t > 0$*,*
- $n \in \mathbb{N}$ *and* $X_0 \in S(H)$ *with* $\operatorname{rank}(X_0) \leq n$*.*

Then there exists a $L_2(H)$*-cylindrical Brownian motion* W *and an adapted stochastic process* $X\colon [0,\infty) \times \Omega \to S_1^+(H)$ *such that* $\forall g, h \in D(A^*), \forall t > 0$:

$$d\langle X_t g, h\rangle_H = \left(n\langle Qg, h\rangle_H + \langle A^* g, X_t h\rangle_H + \langle X_t g, A^* h\rangle_H\right) dt + \langle \sqrt{X}_t\, dW_t \sqrt{Q} g, h\rangle_H + \langle \sqrt{Q}\, dW_t^* \sqrt{X}_t g, h\rangle_H. \tag{Wis_∞}$$

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Moreover, this process is unique in law.

Existence.

Let $Y\colon [0,\infty)\times\Omega \to L(\mathbb{R}^n, H)$ satisfy $Y_0 Y_0^* = X_0$ and

$$Y_t = e^{tA} Y_0 + \int_0^t e^{(t-s)A}\sqrt{Q}\, d\tilde{W}_s$$

where $\tilde{W}$ is an $L_2(\mathbb{R}^n, H)$-cylindrical Brownian motion. Then there exists an $L_2(H)$-cylindrical motion W such that

$$X = YY^*$$

satisfies $\forall g, h \in D(A), \forall t \geq 0$:

$$\begin{aligned} d\langle X_t g, h\rangle_H = {} & (n\langle Qg, h\rangle_H + \langle A^* g, X_t h\rangle_H + \langle X_t g, A^* h\rangle_H)\, dt \\ & + \langle \sqrt{X}_t\, dW_t \sqrt{Q} g, h\rangle_H + \langle \sqrt{Q}\, dW_t^* \sqrt{X}_t g, h\rangle_H. \end{aligned}$$

$\square$

Remark regarding existence

Let

$$\tilde{W}^{(i)} = \tilde{W} e_i, \quad i \in \{1, \dots, n\},$$

where e_i is the i^{th} unit vector in $\mathbb{R}^n$
(recall: $\tilde{W}$ is an $L_2(\mathbb{R}_n, H)$-cylindrical Brownian motion).
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Let $Y^{(1)}, \dots, Y^{(n)} \colon [0, \infty) \times \Omega \to H$ satisfy

$$Y_t^{(i)} = e^{tA} Y_0 e_i + \int_0^t e^{(t-s)A} \sqrt{Q} dW_s^{(i)}, \quad t \geq 0.$$

Then

$$X_t = \sum_{i=1}^n Y_t^{(i)} \otimes Y_t^{(i)}, \quad t \geq 0.$$

Uniqueness in law.

Suppose $(X_t)_{t\geq 0}$ satisfies (Wis$_\infty$). Fix $t > 0$, $u \in S^+(H)$, $v \in S(H)$. By applying Itô's formula to

$$\Phi(s, X_s) = \exp\left(-\operatorname{tr}(\psi(t-s, u-iv)X_s) - n\int_0^{t-s} \operatorname{tr}(\psi(r, u-iv)Q)dr\right)$$

where $\psi\colon [0,t] \times L(H_\mathbb{C}) \to L(H_\mathbb{C})$ is an unknown function, we obtain that if ψ satisfies (Ricc) below then

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$$\begin{cases}\frac{\partial}{\partial t}\psi(t,u-iv)=A^*\psi(t,u-iv)+\psi(t,u-iv)A\\ \qquad -\frac{1}{2}(\psi(t,u-iv)+\psi^T(t,u-iv))Q(\psi(t,u-iv)+\psi^T(t,u-iv)), \quad t\in[0,\infty);\\ \psi(0,u-iv)=u-iv.\end{cases} \tag{Ricc}$$

$\square$

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- We obtain an expression for $\mathbb{E}(e^{-\operatorname{tr}((u-iv)X_t)})$ in each of the following situations
  - $u \in S^+(H)$, $v = 0$;
  - $u = 0$, $v \in S^+(H)$ or $-v \in S^+(H)$;
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- We can replace $n \in \mathbb{N}$ by $\alpha \in \mathbb{R}$.
- Concrete example: for $u \in S^+(H)$ we get

$$\mathbb{E}[\exp(-\operatorname{tr}(uX_t) \,|\, x_0]$$
$$= \det(I_H + 2Q_{t,u})^{-\frac{n}{2}} \exp\left(-\operatorname{tr}\left(e^{tA^*}\sqrt{u}(\mathrm{I}_H + 2Q_{t,u})^{-1}\sqrt{u}e^{tA}x_0\right)\right),$$

with $Q_{t,u} = \sqrt{u}\int_0^t e^{sA}Qe^{sA^*}\,ds\sqrt{u}$.

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Then $\alpha \in \mathbb{N}$ *and* $\operatorname{rank}(X_0) \leq \alpha$. *Moreover,* $\operatorname{rank}(X_t) = \alpha$ *a.s. for almost all* $t \geq 0$.

Proof.

Graczyk, Malecki, and Mayerhofer (2018), Letac and Massam (2018), and Mayerhofer (2019) provide a characterization of finite dimensional Wishart processes via its Laplace transform.

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This ensures $\alpha \in \mathbb{N}$.

To see that $\mathrm{rank}(X_t) = \alpha$ a.s. for almost all $t \geq 0$ we adapt the proof of CFMT (2012) (this requires an extremely technical approximation result). $\square$

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A tractable covariance model that can be of infinite rank:
C., Karbach, and Khedher (2022) establish existence of a process $X\colon [0,\infty)\times\Omega \to S_2^+(H)$ satisfying

$$dX_t = (b + BX_t)\,dt + \,dJ_t^X,$$

where

- $b \in S^+(H)$;
- $B \in \mathcal{L}(L^2(H))$;
- J^X a pure jump process whose compensator ν is an affine function of X; i.e., $\nu(x, d\xi) = m(d\xi) + \langle \frac{\mu(d\xi)}{\|\xi\|^2}, x\rangle$, with $m\colon \mathcal{B}(S^+(H)) \to [0,\infty]$ and $\mu\colon \mathcal{B}(S^+(H)) \to S^+(H)$,

provided b, B, m, and μ satisfy some admissibility conditions.

Open questions and outlook

- Work in progress: equations for the eigenvalues and -vectors of infinite-dimensional Wishart processes (global existence?).

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- Are there Wishart processes that are not of finite rank? (e.g. if Q is not injective)?
- Are there other 'nice' (tractable) stochastic processes taking values in $S_1^+(H)$ (preferably not of finite rank)?

References

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