Infinite-Dimensional Integration (and Function Recovery): Randomized Algorithms and Their Analysis

Michael Gnewuch

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June 30, 2023

Infinite-Variate Integration Problem

Given: $D \neq \emptyset$ domain, μ probability measure on D, class F of input functions $f : D^{\mathbb{N}} \to \mathbb{R}$, and $\mu^{\mathbb{N}} := \bigotimes_{j=1}^{\infty} \mu$ probability measure on $D^{\mathbb{N}}$.

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- Theory: Multivariate integration problems are of interest – infinite-variate integration is limit case when number of variables tends to infinity.
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- impose some structure on the (class of) integrands,
- specify the setting (class of admissable algorithms/cost/error criterion).

Outline

- Infinite-variate Integration
  1. Structure of function spaces
  2. Specification of the setting
  3. Optimal results in a class of function spaces
  4. Extention of optimal results to more general class of function spaces
  5. Further extention of optimal results to different class of function spaces
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Requirement satisfied *iff* H reproducing kernel Hilbert space (RKHS), i.e., if there exists reproducing kernel (RK) $K : X \times X \to \mathbb{R}$ such that

1. $K(\cdot, x) \in H$ for all $x \in X$,
2. $f(x) = \langle f, K(\cdot, x) \rangle_H$ for all $f \in H$, $x \in X$.

Convention: For RK K denote its RKHS by $H(K)$ and norm in $H(K)$ by $\| \cdot \|_K$.

Structure II: Weighted RKHSs, ...

Fix RK $k : D \times D \to \mathbb{R}$ with $\int_D k(x,x)\,\mathrm{d}\mu(x) < \infty$ and $1 \notin H(k)$. Then

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Observation: If $\|f\|_{1+\gamma_j k} \leq 1$, then $\|g\|_k \leq \sqrt{\gamma_j}$.

Structure III: ... Tensor Products, and Function Decomposition

For $u \subset_f \mathbb{N}$ (read as "u is a finite subset of $\mathbb{N}$") put

$$k_u(\mathbf{x}_u, \mathbf{y}_u) := \prod_{j \in u} k(x_j, y_j), \qquad \mathbf{x}_u = (x_j)_{j \in u}, \mathbf{y}_u = (y_j)_{j \in u} \in D^u.$$

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Then

$$H(K_{\boldsymbol{\gamma}}) = \bigotimes_{j \in \mathbb{N}} H(1 + \gamma_j k) = \bigoplus_{u \subset_f \mathbb{N}} H(k_u),$$

resulting in orthogonal function decomposition of $f \in H(K_{\boldsymbol{\gamma}})$

$$f(\mathbf{x}) = \sum_{u \subset_f \mathbb{N}} f_u(\mathbf{x}_u), \quad f_u \in H(k_u), \quad \text{with} \quad \|f\|^2_{K_{\boldsymbol{\gamma}}} = \sum_{u \subset_f \mathbb{N}} \gamma_u^{-1} \|f_u\|^2_{k_u}.$$

Example of FSD: Anchored Decomposition (aka Cut-HDMR)

If there is anchor $a \in D$ such that $k(a,a) = 0$ ("anchor condition"), then decomposition $f = \sum_{u \subset_f \mathbb{N}} f_u$ on $H(K_{\boldsymbol{\gamma}})$ is the anchored decomposition which can be calculated directly as

$$f_u(\mathbf{x}) := \sum_{v \subseteq u} (-1)^{|u \setminus v|} f(\mathbf{x}_v, \mathbf{a}_{\mathbb{N} \setminus v}),$$

see [Kuo, Sloan, Wasilkowski, Woźniakowski '10b].

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Example: Wiener kernel $k : [0,1] \times [0,1] \to \mathbb{R}$, $k(x,y) := \min(x,y)$.

with anchor $a = 0$.

$H(1+k) = \{f \,:\, f, f' \in L^2([0,1])\}$ Sobolev space of order 1

with anchored norm $\|f\|_{1+k}^2 = |f(0)|^2 + \|f'\|_{L^2[0,1]}^2$.

Specifying the Setting I: Admissable Algorithms & Cost Models

Integration functional:

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Deterministic Setting:

Admissable deterministic quadrature formulas:

$$Q_n(f) = \sum_{i=1}^{n} w_i f(\mathbf{x}^{(i)}), \quad \text{where } w_i \in \mathbb{R}, \ \mathbf{x}^{(i)} \in D^{\mathbb{N}}.$$

Cost: Fix default value $a \in D$.

Unrestricted Subspace Sampling [Kuo, Sloan, Wasilkowski, Woźniakowski'10a]

$$\mathrm{cost}(Q_n) := \sum_{i=1}^{n} |\{j \in \mathbb{N} : x_j^{(i)} \neq a\}|$$

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Specifying the Setting II: Error Criterion

Error criterion: (worst case) randomized error of quadrature Q

$$e(Q; K_{\boldsymbol{\gamma}})^2 := \sup_{\|f\|_{K_{\boldsymbol{\gamma}}} \leq 1} E\big((S(f) - Q(f))^2\big)$$

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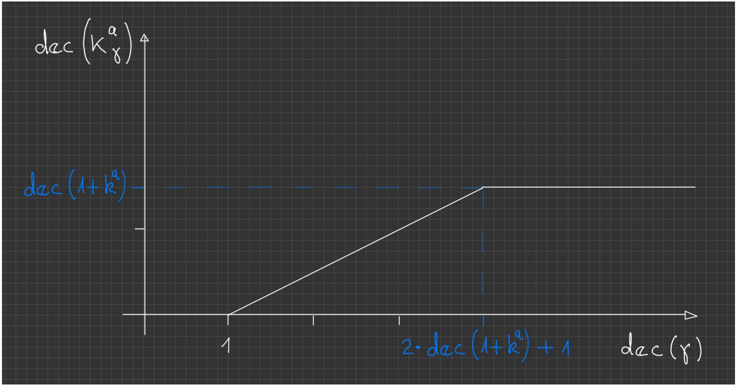
Result 1: Weighted RKHS Based on Anchored Kernel k^a

Theorem 1. *Let $k = k^a$ be RK anchored in $a \in D$ and $K_\gamma = K_\gamma^a$ corresponding weighted RK. Then*

$$\mathrm{dec}(K_\gamma^a) = \min\left(\mathrm{dec}(1 + k^a), \frac{\mathrm{dec}(\gamma) - 1}{2}\right),$$

where $\mathrm{dec}(1 + k^a)$ denotes convergence rate of nth minimal errors of univariate integration on $H(1 + k^a)$.



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Then $H(k_u) = \otimes_{j \in u} H(k)$. Put $\gamma_u := \prod_{j \in u} \gamma_j$ ("product weight").

Structure III: ... Tensor Products, and Function Decomposition

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"Weighted RK":

$$K_{\boldsymbol{\gamma}}(\mathbf{x}, \mathbf{y}) := \prod_{j \in \mathbb{N}} (1 + \gamma_j k(x_j, y_j)) = \sum_{u \subset_f \mathbb{N}} \gamma_u k_u(\mathbf{x}_u, \mathbf{y}_u), \qquad \mathbf{x}, \mathbf{y} \in D^{\mathbb{N}}.$$

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Then

$$H(K_{\boldsymbol{\gamma}}) = \bigotimes_{j\in\mathbb{N}} H(1 + \gamma_j k) = \bigoplus_{u\subset_f \mathbb{N}} H(k_u),$$

resulting in orthogonal function decomposition of $f \in H(K_{\boldsymbol{\gamma}})$

$$f(\mathbf{x}) = \sum_{u\subset_f \mathbb{N}} f_u(\mathbf{x}_u), \quad f_u \in H(k_u), \quad \text{with} \quad \|f\|^2_{K_{\boldsymbol{\gamma}}} = \sum_{u\subset_f \mathbb{N}} \gamma_u^{-1} \|f_u\|^2_{k_u}.$$

Example of FSD: Anchored Decomposition (aka Cut-HDMR)

If there is anchor $a \in D$ such that $k(a,a) = 0$ ("anchor condition"), then decomposition $f = \sum_{u \subset_f \mathbb{N}} f_u$ on $H(K_{\boldsymbol{\gamma}})$ is the anchored decomposition which can be calculated directly as

$$f_u(\mathbf{x}) := \sum_{v \subseteq u} (-1)^{|u \setminus v|} f(\mathbf{x}_v, \mathbf{a}_{\mathbb{N} \setminus v}),$$

see [Kuo, Sloan, Wasilkowski, Woźniakowski '10b].

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Example: Wiener kernel $k : [0,1] \times [0,1] \to \mathbb{R}$, $k(x,y) := \min(x,y)$.

with anchor $a = 0$.

$H(1+k) = \{f \,:\, f, f' \in L^2([0,1])\}$ Sobolev space of order 1

with anchored norm $\|f\|_{1+k}^2 = |f(0)|^2 + \|f'\|_{L^2[0,1]}^2$.

Specifying the Setting I: Admissable Algorithms & Cost Models

Integration functional:

$$S(f) = \int_{D^{\mathbb{N}}} f(\mathbf{x}) \, \mathrm{d}\mu^{\mathbb{N}}(\mathbf{x})$$

Deterministic Setting:

Admissable deterministic quadrature formulas:

$$Q_n(f) = \sum_{i=1}^{n} w_i f(\mathbf{x}^{(i)}), \quad \text{where } w_i \in \mathbb{R}, \ \mathbf{x}^{(i)} \in D^{\mathbb{N}}.$$

Cost: Fix default value $a \in D$.

Unrestricted Subspace Sampling [Kuo, Sloan, Wasilkowski, Woźniakowski'10a]

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Error criterion: (worst case) randomized error of quadrature Q

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"(Polynomial) Convergence rate" of decreasing null sequence $\mathbf{s} := (s_n)_{n\in\mathbb{N}}$:

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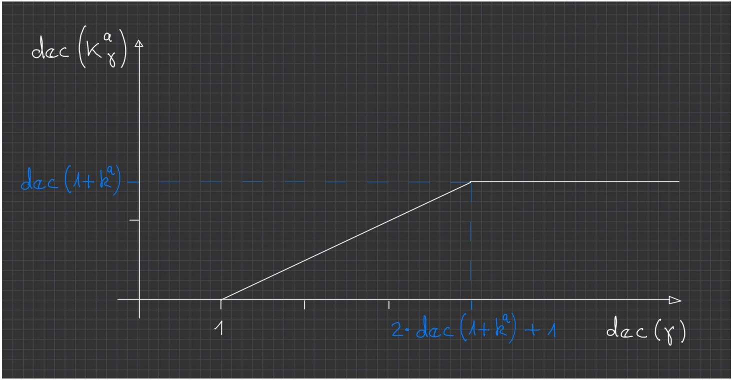
Result 1: Weighted RKHS Based on Anchored Kernel k^a

Theorem 1. *Let* $k = k^a$ *be RK anchored in* $a \in D$ *and* $K_{\boldsymbol{\gamma}} = K^a_{\boldsymbol{\gamma}}$ *corresponding weighted RK. Then*

$$\mathrm{dec}(K^a_{\boldsymbol{\gamma}}) = \min\left(\mathrm{dec}(1 + k^a), \frac{\mathrm{dec}(\boldsymbol{\gamma}) - 1}{2}\right),$$

where $\mathrm{dec}(1 + k^a)$ *denotes convergence rate of* n*th minimal errors of univariate integration on* $H(1 + k^a)$.



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Theorem 1. *Let $k = k^a$ be RK anchored in $a \in D$ and $K_{\boldsymbol{\gamma}} = K_{\boldsymbol{\gamma}}^a$ corresponding weighted RK. Then*

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Contributions to proof of Theorem 1:

- Lower error bounds: Kuo, Sloan, Wasilkowski & Woźniakowski'10a (deterministic case); G.'13 (randomized case)
- Upper error bounds: Plaskota & Wasilkowski'11 (via multivariate decomposition method (MDM))

  Earlier partial results in deterministic case: Kuo, Sloan, Wasilkowski & Woźniakowski'10a (via MDM); Niu, Hickernell, Müller-Gronbach & Ritter'11; G.'12 (both via multilevel algorithms)

Multivariate Decomposition Method

Recall: k^a RK anchored in $a \in D$ and $K^a_{\boldsymbol{\gamma}}$ corresponding weighted RK.
Anchored decomposition for $f \in H(K^a_{\boldsymbol{\gamma}})$: $f = \sum_{u \subset_f \mathbb{N}} f_u$, $f_u \in H(k_u)$.

Observation: S continuous on $H(K^a_{\boldsymbol{\gamma}})$, hence

$$S(f) = \int_{D^{\mathbb{N}}} f(\mathbf{x}) \, \mathrm{d}\mu(\mathbf{x}) = \sum_{u \subset_f \mathbb{N}} \int_{D^u} f_u(\mathbf{x}_u) \, \mathrm{d}\mu^u(\mathbf{x}_u).$$

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Multivariate decomposition method (MDM):

1. Choose finite set $\mathcal{A}$ ("set of active variables") of most important groups of variables.
2. Choose for each $u \in \mathcal{A}$ quadrature Q_{u,n_u} using n_u samples to approximate $\int_{D^u} f_u(\mathbf{x}_u) \,\mathrm{d}\mu^u(\mathbf{x}_u)$.

Final algorithm Q^{MDM} is of form

$$Q^{\mathrm{MDM}}(f) = \sum_{u \in \mathcal{A}} Q_{u,n_u}(f_u)$$

Multivariate Decomposition Method

MDM *fka* changing dimension algorithm for ∞-variate setting introduced in

- Kuo, Sloan, Wasilkowski &Woźniakowski'10; Plaskota & Wasilkowski'11 (∞-variate integration)
- Wasilkowski & Woźniakowski'11a, '11b (∞-variate approximation).

Further papers on MDM include: Wasilkowski'12; G.'13; Dick & G.'14a, 14b; Plaskota & Wasilkowski'14; Wasilkowski'14; Gilbert & Wasilkowski'17; G., Hefter, Hinrichs & Ritter'17; G., Hefter, Hinrichs, Ritter & Wasilkowski'17; Kuo, Nuyens, Plaskota, Sloan, Wasilkowski'17; Gilbert, Kuo, Nuyens & Wasilkowski'18; G., Hefter, Hinrichs, Ritter & Wasilkowski'19; G. & Wnuk'20; G., Hinrichs, Ritter & Rüßmann'23+

Similar idea used for multivariate integration in Griebel & Holtz'10 (dimension-wise quadrature methods).

How to choose building blocks Q_{u,n_u} for MDM?

1. Choose sequence $(Q_k)_{k\in\mathbb{N}}$ of quadratures for univariate integration on $H(1+k^a)$.
2. Employ Smolyak's method to obtain Q_{u,n_u}: For $f_u \in H(k_u^a)$

$$Q_{u,n_u} f_u = \left(\sum_{L-|u|+1 \le |\ell|_1 \le L} (-1)^{L-|\ell|_1} \binom{|u|-1}{L-|\ell|_1} \otimes_{j\in u} Q_{\ell_j} \right) f_u .$$

Smolyak's method (*aka* sparse grid method, etc.) is "universal method" for approximation of tensor product problems, see, e.g.,

- Deterministic Case: [Smolyak'63; Temlyakov'87; Baszenski & Delvos'93; Wasilkowski & Woźniakowski'95; Novak & Ritter'96; Gerstner, Griebel'98;...
  Surveys: [Bungartz, Griebel'04; Novak & Woźniakowski'08; Dũng, Temlyakov &T. Ullrich'16; Tempone & Wolfers'18;...]
- Randomized Case: [Dick, Leobacher & Pillichshammer'07; Heinrich & Milla'11; G. & Wnuk'20; Wnuk & G.'21].

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Examples of Non-Anchored RKHSs

The following RKHSs are not based on an anchored kernel $k = k^a$:

Spaces defined via ...

... derivatives:

- Sobolev spaces with classical norms,
- unanchored Sobolev spaces,
- ...

... Fourier coefficients w.r.t. orthonormal basis of L^2 (including 1):

- Korobov spaces,
- Walsh spaces,
- Haar wavelet spaces,
- Hermite spaces,
- cosine spaces,
- ...

Result 2: Weighted RKHS Based on General RK k

Theorem 2 [G., Hefter, Hinrichs & Ritter'17; G. & Wnuk'20][1] . *Let k be general RK and K_{γ} corresponding weighted RK. Then*

$$\mathrm{dec}(K_{\gamma}) = \min\left(\mathrm{dec}(1+k), \frac{\mathrm{dec}(\gamma)-1}{2}\right),$$

where $\mathrm{dec}(1+k)$ denotes convergence rate of Nth minimal errors of univariate integration on $H(1+k)$.

[1] Results for ANOVA kernels in randomized setting were proved earlier in [Hickernell, Müller-Gronbach, Niu & Ritter'10], [Baldeaux & G. 14], [Dick & G.'14b].

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How to prove the more general Theorem 2?

Simple observation: For Banach spaces $X \subseteq Y$ of integrable functions:

$$X \hookrightarrow Y$$

with embedding constant C implies

$$e(Q,X) \le Ce(Q,Y) \quad \text{for all quadratures } Q,$$

resulting in

$$e(n,X) \le Ce(n,Y) \quad \text{and} \quad \mathrm{dec}(X) \ge \mathrm{dec}(Y).$$

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"Universality of Anchored Spaces"

Construction of associated anchored RKHS of $H(1+k)$

Subspace $\{f \in H(1+k) \mid f(a) = 0\}$ of RKHS $H(1+k)$ has RK

$$k^a : D \times D \to \mathbb{R} \quad \text{with} \quad k^a(a,a) = 0$$

and $H(1+k^a) = H(1+k)$ as vector spaces with equivalent norms, hence $\mathrm{dec}(1+k^a) = \mathrm{dec}(1+k)$.

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With associated anchored kernel k^a of k again construct weighted space of ∞-variate functions $H(K^a_{\boldsymbol{\gamma}})$:

$$K^a_{\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y}) = \prod_{j\in\mathbb{N}} (1 + \gamma_j k^a(x_j, y_j)), \qquad \mathbf{x},\mathbf{y} \in D^{\mathbb{N}}.$$

“Embedding Triple”

In general, $H(K_{\boldsymbol{\gamma}}) \not\subseteq H(K_{\boldsymbol{\gamma}}^{a})$ and $H(K_{\boldsymbol{\gamma}}^{a}) \not\subseteq H(K_{\boldsymbol{\gamma}})$ [Hefter & Ritter '14].

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Remedy: Manipulate the Weights!

For $c > 0$ let $c\boldsymbol{\gamma} := (c\gamma_j)_{j\in\mathbb{N}}$. Note that $\mathrm{dec}(c\boldsymbol{\gamma}) = \mathrm{dec}(\boldsymbol{\gamma})$.

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Proposition [G., Hefter, Hinrichs & Ritter'17]. *There exists $0 < c_0 < 1$ such that*

$$H(K^a_{c_0\boldsymbol{\gamma}}) \hookrightarrow H(K_{\boldsymbol{\gamma}}) \hookrightarrow H(K^a_{c_0^{-1}\boldsymbol{\gamma}}).$$

In general, $H(K^a_{c_0\boldsymbol{\gamma}}) \subsetneq H(K_{\boldsymbol{\gamma}}) \subsetneq H(K^a_{c_0^{-1}\boldsymbol{\gamma}})$.

Structure IV: Spaces of Increasing Smoothness

Let $(e_\nu)_{\nu \in \mathbb{N}_0}$ ONB of $L^2(D, \mu)$ with $e_0 = 1$.

Consider "Fourier weights" $\alpha_{\nu,j} \geq 1\,,\ \nu, j \in \mathbb{N}$, with

(A 1) $\alpha_{\nu,j} \leq \alpha_{\nu+1,j}$ and $\alpha_{\nu,j} \leq \alpha_{\nu,j+1}$ for all $\nu, j \in \mathbb{N}$,

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For each variable $j \in \mathbb{N}$ define RK

$$k_j(x,y) = \sum_{\nu\in\mathbb{N}} \alpha_{\nu,j}^{-1} \cdot e_\nu(x) \cdot e_\nu(y), \qquad x, y \in \mathbb{R};$$

Then

$$H(1+k_j) = \Big\{ f \in L^2(D,\mu) \,:\, \sum_{\nu\in\mathbb{N}} \alpha_{\nu,j} \, \langle f, e_\nu \rangle_{L^2}^2 < \infty \Big\}.$$

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Examples: Korobov spaces, Walsh spaces, Haar spaces, Hermite spaces,....

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Build "increasing smoothness RK"

$$K(\mathbf{x},\mathbf{y}) := \prod_{j\in\mathbb{N}} (1 + k_j(x_j, y_j)), \qquad \mathbf{x}, \mathbf{y} \in D^{\mathbb{N}}.$$

Result 3: Spaces of Increasing Smoothness

Define weights

$$\gamma_j := \sup_{\nu \in \mathbb{N}} \frac{\alpha_{\nu,1}}{\alpha_{\nu,j}}, \qquad j \in \mathbb{N}.$$

Interpretation of weights: $\sqrt{\gamma_j}$ norm of embedding $H(k_j) \hookrightarrow H(k_1)$.

Sequence $\boldsymbol{\gamma} = (\gamma_j)_{j \in \mathbb{N}}$ monotone decreasing. Assume $\sum_{j \in \mathbb{N}} \gamma_j < \infty$.

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Theorem 3 [G., Hefter, Hinrichs, Ritter & Wasilkowski'19; G., Hinrichs, Ritter & Rüßmann'23+].

$$\min\left(\operatorname{dec}(1+k_1)\,,\ \frac{\operatorname{dec}(\boldsymbol{\gamma})-1}{2}\right) \le \operatorname{dec}(K)$$

$$\le \min\left(\operatorname{dec}(1+k_1)\,,\ \frac{\operatorname{dec}((\alpha_{1,j}^{-1})_{j\in\mathbb{N}})-1}{2}\right)$$

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Examples: Let $1 < r_1 \leq r_2 \leq r_3 \leq \ldots$

Fourier weights of ...

- ... polynomial growth (PG): $\alpha_{\nu,j} := (\nu+1)^{r_j}$,
- ... exponential growth (EG): $\alpha_{\nu,j} := 2^{r_j\cdot\nu}$.

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Fourier weights of ...

- ... polynomial growth (PG): $\alpha_{\nu,j} := (\nu+1)^{r_j}$,
- ... exponential growth (EG): $\alpha_{\nu,j} := 2^{r_j \cdot \nu}$.

For both cases (PG) & (EG): $\alpha_{1,j}^{-1} = 2^{-r_j}$ and $\gamma_j = 2^{r_1 - r_j}$, hence

$$\operatorname{dec}(\boldsymbol{\gamma}) = \operatorname{dec}((\alpha_{1,j}^{-1})_{j\in\mathbb{N}}) = \liminf_{j\to\infty}\left(\frac{\ln(2)}{\ln(j)}\, r_j\right),$$

resulting in matching bounds in Theorem 3.

L^2-Approximation

Solution Operator: Embedding operator $S : H(K) \to L^2(D^{\mathbb{N}}, \mu^{\mathbb{N}})$, $f \mapsto f$.

Algorithms: Linear algorithms $A : H(K) \to \mathbb{R}$ of form

$$A(f) := \sum_{i=1}^{m} w_i \cdot f\big(\mathbf{x}^{(i)}\big) \tag{3}$$

where $m \in \mathbb{N}$, $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)} \in D^{\mathbb{N}}$, and $w_1, \ldots, w_m \in L^2(D^{\mathbb{N}}, \mu^{\mathbb{N}})$.

Cost: We only account for function evaluations, fix nominal value $a \in \mathbb{R}$, ...

Error Criterion: Worst case randomized/deterministic error of algorithm A ...

Results: Theorem 1, 2, and 3 hold verbatim[2]! Again, upper error bounds are established via MDM based on Smolyak algorithms.

[2] But the univariate convergence rates of n-th minimal errors of integration and L^2-approximation may differ.

Summary

- Infinite-variate Integration:
  1. On spaces of integrands we first imposed structure (reproducing kernels, tensor product structure, weights, anchored function decomposition).
  2. Afterwards we fixed the setting (admissable algorithms, costs, and error criterion). In that setting...
  3. ... MDM achieved optimal results on anchored weighted RKHSs by exploiting the anchored function decomposition.
  4. ... an Embedding framework was used to prove that MDM achieve also optimal results on general weighted RKHSs.
  5. ... another embedding strategy was used to prove that MDM achieve also optimal results on RKHSs of increasing smoothness.
- Infinite-variate L^2-Approximation:
  Same holds for L^2-approximation (main results Theorem 1, 2 & 3 for integration hold verbatim for L^2-approximation).
- Concluding Remark: Our embedding techniques allow to transfer error bounds for linear algorithms for multi- or ∞-variate linear problems on one RKHS of weighted type or of increasing smoothness to different ones.

Sketch of Proof of Theorem 3

Consider *new* weighted RKs

$$k_j^{\mathrm{up}}(x,y) := \gamma_j\, k_1(x,y), \quad \text{and} \quad k_j^{\mathrm{low}}(x,y) := \alpha_{1,j}^{-1}\, e_1(x)e_1(y),$$

and put

$$K_{\boldsymbol{\gamma}}^{\mathrm{up}}(\mathbf{x},\mathbf{y}) := \prod_{j=1}^{\infty}(1+k_j^{\mathrm{up}}(x_j,y_j)), \quad \text{and} \quad K_{\alpha_{1,j}^{-1}}^{\mathrm{low}}(\mathbf{x},\mathbf{y}) := \prod_{j=1}^{\infty}(1+k_j^{\mathrm{low}}(x_j,y_j)).$$

Then

$$H(K_{\alpha_{1,j}^{-1}}^{\mathrm{low}}) \hookrightarrow H(K) \hookrightarrow H(K_{\boldsymbol{\gamma}}^{\mathrm{up}}).$$

Since $\mathrm{dec}(1+k_1^{\mathrm{up}}) = \mathrm{dec}(1+k_1)$ and $\mathrm{dec}(1+k_1) \geq \mathrm{dec}(K)$, we obtain

$$\min\left(\mathrm{dec}(1+k_1)\,,\ \frac{\mathrm{dec}(\boldsymbol{\gamma})-1}{2}\right) \leq \mathrm{dec}(K)$$
$$\leq \min\left(\mathrm{dec}(1+k_1)\,,\ \frac{\mathrm{dec}((\alpha_{1,j}^{-1})_{j\in\mathbb{N}})-1}{2}\right)$$

Observation: $\dim(H(1+k_j^{\mathrm{low}})) = 2$, while $\dim(H(1+k_j)) = \infty$ and typically $H(1+k_{j+1}) \hookrightarrow H(1+k_j)$ compactly, while $H(1+k_{j+1}^{\mathrm{up}}) = H(1+k_j^{\mathrm{up}})$ with equivalent norms.

An Alternative Cost Model

Randomized Quadrature: $Q_n(f) = \sum_{i=1}^{n} w_i f(\mathbf{x}^{(i)})$, where $w_i \in \mathbb{R}$, $\mathbf{x}^{(i)} \in D^{v_i}$.

Unrestricted Subspace Sampling [Kuo, Sloan, Wasilkowski, Woźniakowski'10a]

$$\mathrm{cost}_{\mathrm{unr}}(Q_n) := E\left[\sum_{i=1}^{n} \left|\{j \in \mathbb{N} \,:\, x_j^{(i)} \neq a\}\right|\right]$$

Nested Subspace Sampling [Creutzig, Dereich, Müller-Gronbach, Ritter'09]

$$\mathrm{cost}_{\mathrm{nest}}(Q_n) := E\left[\sum_{i=1}^{n} \max\{j \in \mathbb{N} \,:\, x_j^{(i)} \neq a\}\right]$$

"Rule of Thumb": For USS use multivariate decomposition method (MDM), but for NSS use multilevel algorithms!

General references for multilevel algorithms: Heinrich '98, '01 (integral equations/parametric integration); Giles '08a, '08b; Giles, Higham & Mao (SDEs); Giles & Waterhouse '09 (QMC-ML); ... *now better look at* http://people.maths.ox.ac.uk/gilesm/mlmc_community.html

Multilevel for ∞-dimensional integration problem: Hickernell, Müller-Gronbach, Niu & Ritter '10; Niu, Hickernell, Müller-Gronbach & Ritter '11; G. '12; Baldeaux '12; G.'13; Baldeaux & G. '14; Dick & G. '14a; Hefter '14; G., Hefter, Hinrichs & Ritter '17.

Comparison: Multilevel Algorithms and MDM

QMC-Multilevel Algorithm: $Q_L^{\mathrm{ML}}(f) := \sum_{\ell=1}^{L} Q_\ell(f)$, where

$$Q_\ell(f) := \frac{1}{n_\ell} \sum_{i=1}^{n_\ell} \Big(f(x_1^{(\ell,i)}, \ldots, x_{2^\ell}^{(\ell,i)}, a, a, \ldots) - f(x_1^{(\ell,i)}, \ldots, x_{2^{\ell-1}}^{(\ell,i)}, a, a, \ldots) \Big)$$

Let $f \in H(K_\gamma^a)$ with anchored function decomposition $f = \sum_{u \subset_f \mathbb{N}} f_u$.

Solution operator:

$$S(f) = \int_{D^{\mathbb{N}}} f(\mathbf{x}) \, \mathrm{d}\mu^{\mathbb{N}}(\mathbf{x}) = \sum_{u \subset_f \mathbb{N}} \int_{D^u} f_u(\mathbf{x}_u) \, \mathrm{d}\mu^u(\mathbf{x}_u)$$

Multivariate decomposition method:

$$Q^{\mathrm{MDM}}(f) = \sum_{u \subset_f \mathbb{N}} Q_{u,n_u}(f_u)$$

Multilevel algorithm:

$$Q^{\mathrm{ML}}(f) = \sum_{\ell=1}^{L} \Bigg(\sum_{\substack{u \subseteq \{1,2,\ldots,2^\ell\} \\ u \not\subseteq \{1,2,\ldots,2^{\ell-1}\}}} Q_\ell(f_u) \Bigg)$$

Comparison: Multilevel Algorithms and MDM

QMC-Multilevel Algorithm: $Q_L^{\mathrm{ML}}(f) := \sum_{\ell=1}^{L} Q_\ell(f)$, where

$$Q_\ell(f) := \frac{1}{n_\ell} \sum_{i=1}^{n_\ell} \Big(f(x_1^{(\ell,i)}, \ldots, x_{2^\ell}^{(\ell,i)}, a, a, \ldots) - f(x_1^{(\ell,i)}, \ldots, x_{2^{\ell-1}}^{(\ell,i)}, a, a, \ldots) \Big)$$

Cost: $$\mathrm{cost}_{\mathrm{nest}}(Q_L^{\mathrm{ML}}) = \mathrm{cost}_{\mathrm{unr}}(Q_L^{\mathrm{ML}}) \le \sum_{\ell=1}^{L} 2\, n_\ell\, 2^\ell$$

Multivariate decomposition method: Q^{MDM}:

$$Q^{\mathrm{MDM}}(f) = \sum_{u \in \mathcal{A}_\varepsilon} Q_{u,n_u}(f_u)$$

$f_u(\mathbf{x}) = \sum_{v \subseteq u} (-1)^{|u \setminus v|} f(\mathbf{x}_v)$

Cost: $$\mathrm{cost}_{\mathrm{unr}}(Q_L^{\mathrm{MDM}}) = O\left(\sum_{u \in \mathcal{A}_\varepsilon} n_u 2^{|u|} |u| \right)$$

(Recall: $f_u(\mathbf{x}) = \sum_{v \subseteq u} (-1)^{|u \setminus v|} f(\mathbf{x}_v)$.
Cost for evaluating f_u in unrestricted model: $O(2^{|u|}|u|)$.
"Typically": $u \in \mathcal{A}_\varepsilon \implies |u| = o(\ln(1/\varepsilon))$.)

Example of Non-Anchored Decomposition: ANOVA Decomp.

ANOVA condition $\int_D k(x,y)\,d\mu(y) = 0 \;\; \forall\, x \in D$ implies that function decomp. $f = \sum_{u \subset_f \mathbb{N}} f_u$ in $H(K_{\boldsymbol{\gamma}})$ is ANOVA decomp. (in $L^2(D^{\mathbb{N}}, \mu^{\mathbb{N}})$):

$$f_{\emptyset}(\mathbf{x}) := \int_{D^{\mathbb{N}}} f(\mathbf{y})\, \mathrm{d}\mu^{\mathbb{N}}(\mathbf{y})\,,$$
$$f_u(\mathbf{x}) := \int_{D^{\mathbb{N}\setminus u}} f(\mathbf{x}_u, \mathbf{y}_{\mathbb{N}\setminus u})\, \mathrm{d}\mu^{\mathbb{N}\setminus u}(\mathbf{y}_{\mathbb{N}\setminus u}) - \sum_{v \subsetneq u} f_v(\mathbf{x})\,,$$

where $\mathbf{x}_u := (x_j)_{j\in u}$, $\mathbf{y}_{\mathbb{N}\setminus u} := (y_j)_{j \in \mathbb{N}\setminus u}$.

This implies

$$\|f\|^2_{L^2(D^{\mathbb{N}},\mu^{\mathbb{N}})} = \sum_{u \subset_f \mathbb{N}} \|f_u\|^2_{L^2(D^u,\mu^u)} \quad \text{and} \quad \mathrm{Var}(f) = \sum_{u \subset_f \mathbb{N}} \mathrm{Var}(f_u).$$

(∞-variate ANOVA decomposition was studied in:
Hickernell, Müller-Gronbach, Niu, Ritter '10; G., Baldeaux '14; Dick, G. '14b; Griebel, Kuo, Sloan '16;...)

Some Motivation

Given: $X = (X_t)_{t\in[0,1]}$ stochastic process with series expansion

$$X_t = \sum_{j=1}^{\infty} \xi_j e_j(t),$$

$e_1, e_2, \ldots$ deterministic functions on $[0,1]$, $\xi_1, \xi_2, \ldots$ i.i.d. r.v. with distribution μ on D and $E[\xi_i] = 0$.

Examples: Consider $D = \mathbb{R}$ and $\mu = N(0,1)$.

- Karhunen-Loéve expansion of zero-mean Gaussian process X; then $(e_j)_{j\in\mathbb{N}}$ orthogonal system in $L^2([0,1])$ with $\sum_{j=1}^{\infty} \|e_j\|^2_{L^2([0,1])} < \infty$.
- Lévy-Ciesielski or Brownian Bridge construction of Brownian motion X; then $e_1, e_2, \ldots$ Schauder functions.

Task: For given functional φ on path space of X evaluate $E[\varphi(X)]$.

Observation:

$$E[\varphi(X)] = \int_{D^{\mathbb{N}}} f(\mathbf{x}) \, \mathrm{d}\mu^{\mathbb{N}}(\mathbf{x}), \quad \text{where} \quad f(\mathbf{x}) = \varphi\left(\sum_{j=1}^{\infty} x_j e_j\right).$$

Computational Approach

Step 1: "Truncation":

$$\sum_{j=1}^{\infty} \xi_j e_j \approx \sum_{j=1}^{d} \xi_j e_j \quad \text{and} \quad E[\varphi(X)] \approx E\left[\varphi\left(\sum_{j=1}^{d} \xi_j e_j\right)\right]$$

Step 2: Approximate

$$E\left[\varphi\left(\sum_{j=1}^{d} \xi_j e_j\right)\right] = \int_{D^{\mathbb{N}}} f(x_1, \dots, x_d, 0, 0, \dots)\, \mathrm{d}\mu^{\mathbb{N}}(\mathbf{x})$$
$$= \int_{D^d} f(x_1, \dots, x_d, 0, 0, \dots)\, \mathrm{d}\mu^{d}(x_1, \dots, x_d)$$

with algorithms for multivariate integration (recall $f(\mathbf{x}) = \varphi(\sum_{j=1}^{\infty} x_j e_j)$).

Result 1: Weighted RKHS Based on Anchored Kernel k^a

Theorem 1. *Let $k = k^a$ be RK anchored in $a \in D$ and $K_{\boldsymbol{\gamma}} = K^a_{\boldsymbol{\gamma}}$ corresponding weighted RK. Then*

$$\mathrm{dec}(K^a_{\boldsymbol{\gamma}}) = \min\left(\mathrm{dec}(1+k^a), \frac{\mathrm{dec}(\boldsymbol{\gamma})-1}{2}\right),$$

where $\mathrm{dec}(1+k^a)$ denotes convergence rate of nth minimal errors of univariate integration on $H(1+k^a)$.

Contributions to proof of Theorem 1:

- Lower error bounds: Kuo, Sloan, Wasilkowski & Woźniakowski'10a (deterministic case); G.'13 (randomized case)
- Upper error bounds: Plaskota & Wasilkowski'11 (via multivariate decomposition method (MDM))

  Earlier partial results in deterministic case: Kuo, Sloan, Wasilkowski & Woźniakowski'10a (via MDM); Niu, Hickernell, Müller-Gronbach & Ritter'11; G.'12 (both via multilevel algorithms)

Multivariate Decomposition Method

Recall: k^a RK anchored in $a \in D$ and $K^a_{\boldsymbol{\gamma}}$ corresponding weighted RK.
Anchored decomposition for $f \in H(K^a_{\boldsymbol{\gamma}})$: $f = \sum_{u \subset_f \mathbb{N}} f_u$, $f_u \in H(k_u)$.

Observation: S continuous on $H(K^a_{\boldsymbol{\gamma}})$, hence

$$S(f) = \int_{D^{\mathbb{N}}} f(\mathbf{x}) \, \mathrm{d}\mu(\mathbf{x}) = \sum_{u \subset_f \mathbb{N}} \int_{D^u} f_u(\mathbf{x}_u) \, \mathrm{d}\mu^u(\mathbf{x}_u).$$

Multivariate Decomposition Method

Recall: k^a RK anchored in $a \in D$ and $K_{\boldsymbol{\gamma}}^a$ corresponding weighted RK.
Anchored decomposition for $f \in H(K_{\boldsymbol{\gamma}}^a)$: $f = \sum_{u \subset_f \mathbb{N}} f_u$, $f_u \in H(k_u)$.

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$$S(f) = \int_{D^{\mathbb{N}}} f(\mathbf{x}) \, \mathrm{d}\mu(\mathbf{x}) = \sum_{u \subset_f \mathbb{N}} \int_{D^u} f_u(\mathbf{x}_u) \, \mathrm{d}\mu^u(\mathbf{x}_u).$$

Multivariate decomposition method (MDM):

1. Choose finite set $\mathcal{A}$ ("set of active variables") of most important groups of variables.
2. Choose for each $u \in \mathcal{A}$ quadrature Q_{u,n_u} using n_u samples to approximate $\int_{D^u} f_u(\mathbf{x}_u) \, \mathrm{d}\mu^u(\mathbf{x}_u)$.

Final algorithm Q^{MDM} is of form

$$Q^{\mathrm{MDM}}(f) = \sum_{u \in \mathcal{A}} Q_{u,n_u}(f_u)$$

Multivariate Decomposition Method

MDM *fka* changing dimension algorithm for ∞-variate setting introduced in

- Kuo, Sloan, Wasilkowski &Woźniakowski'10; Plaskota & Wasilkowski'11 (∞-variate integration)
- Wasilkowski & Woźniakowski'11a, '11b (∞-variate approximation).

Further papers on MDM include: Wasilkowski'12; G.'13; Dick & G.'14a, 14b; Plaskota & Wasilkowski'14; Wasilkowski'14; Gilbert & Wasilkowski'17; G., Hefter, Hinrichs & Ritter'17; G., Hefter, Hinrichs, Ritter & Wasilkowski'17; Kuo, Nuyens, Plaskota, Sloan, Wasilkowski'17; Gilbert, Kuo, Nuyens & Wasilkowski'18; G., Hefter, Hinrichs, Ritter & Wasilkowski'19; G. & Wnuk'20; G., Hinrichs, Ritter & Rüßmann'23+

Similar idea used for multivariate integration in Griebel & Holtz'10 (dimension-wise quadrature methods).

How to choose building blocks Q_{u,n_u} for MDM?

1. Choose sequence $(Q_k)_{k\in\mathbb{N}}$ of quadratures for univariate integration on $H(1+k^a)$.
2. Employ Smolyak's method to obtain Q_{u,n_u}: For $f_u \in H(k_u^a)$

$$Q_{u,n_u} f_u = \left(\sum_{L-|u|+1 \le |\ell|_1 \le L} (-1)^{L-|\ell|_1} \binom{|u|-1}{L-|\ell|_1} \otimes_{j\in u} Q_{\ell_j} \right) f_u.$$

Smolyak's method (*aka* sparse grid method, etc.) is "universal method" for approximation of tensor product problems, see, e.g.,

- Deterministic Case: [Smolyak'63; Temlyakov'87; Baszenski & Delvos'93; Wasilkowski & Woźniakowski'95; Novak & Ritter'96; Gerstner, Griebel'98;...
  Surveys: [Bungartz, Griebel'04; Novak & Woźniakowski'08; Dũng, Temlyakov &T. Ullrich'16; Tempone & Wolfers'18;...]
- Randomized Case: [Dick, Leobacher & Pillichshammer'07; Heinrich & Milla'11; G. & Wnuk'20; Wnuk & G.'21].

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- Randomized Case: [Dick, Leobacher & Pillichshammer'07; Heinrich & Milla'11; G. & Wnuk'20; Wnuk & G.'21].

Examples of Non-Anchored RKHSs

The following RKHSs are not based on an anchored kernel $k = k^a$:

Spaces defined via ...

... derivatives:

- Sobolev spaces with classical norms,
- unanchored Sobolev spaces,
- ...

... Fourier coefficients w.r.t. orthonormal basis of L^2 (including 1):

- Korobov spaces,
- Walsh spaces,
- Haar wavelet spaces,
- Hermite spaces,
- cosine spaces,
- ...

Result 2: Weighted RKHS Based on General RK k

Theorem 2 [G., Hefter, Hinrichs & Ritter'17; G. & Wnuk'20]. *Let k be general RK and $K_{\boldsymbol{\gamma}}$ corresponding weighted RK. Then*

$$\operatorname{dec}(K_{\boldsymbol{\gamma}}) = \min\left(\operatorname{dec}(1+k), \frac{\operatorname{dec}(\boldsymbol{\gamma})-1}{2}\right),$$

where $\operatorname{dec}(1+k)$ denotes convergence rate of Nth minimal errors of univariate integration on $H(1+k)$.

Result 2: Weighted RKHS Based on General RK k

Theorem 2 [G., Hefter, Hinrichs & Ritter'17; G. & Wnuk'20]. *Let k be general RK and K_γ corresponding weighted RK. Then*

$$\mathrm{dec}(K_{\boldsymbol{\gamma}}) = \min\left(\mathrm{dec}(1+k), \frac{\mathrm{dec}(\boldsymbol{\gamma})-1}{2}\right),$$

where $\mathrm{dec}(1+k)$ denotes convergence rate of Nth minimal errors of univariate integration on $H(1+k)$.

How to prove the more general Theorem 2?

Simple observation: For Banach spaces $X \subseteq Y$ of integrable functions:

$$X \hookrightarrow Y$$

with embedding constant C implies

$$e(Q,X) \le C e(Q,Y) \quad \text{for all quadratures } Q,$$

resulting in

$$e(n,X) \le C e(n,Y) \qquad \text{and} \qquad \mathrm{dec}(X) \ge \mathrm{dec}(Y).$$

"Universality of Anchored Spaces"

Construction of associated anchored RKHS of $H(1+k)$

Subspace $\{f \in H(1+k) \mid f(a) = 0\}$ of RKHS $H(1+k)$ has RK

$$k^a : D \times D \to \mathbb{R} \quad \text{with} \quad k^a(a,a) = 0$$

and $H(1+k^a) = H(1+k)$ as vector spaces with equivalent norms, hence $\mathrm{dec}(1+k^a) = \mathrm{dec}(1+k)$.

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and $H(1+k^a) = H(1+k)$ as vector spaces with equivalent norms, hence $\mathrm{dec}(1+k^a) = \mathrm{dec}(1+k)$.

With associated anchored kernel k^a of k again construct weighted space of ∞-variate functions $H(K_{\boldsymbol{\gamma}}^a)$:

$$K_{\boldsymbol{\gamma}}^a(\mathbf{x},\mathbf{y}) = \prod_{j\in\mathbb{N}} (1 + \gamma_j k^a(x_j, y_j)), \qquad \mathbf{x},\mathbf{y} \in D^{\mathbb{N}}.$$

"Embedding Triple"

In general, $H(K_{\boldsymbol{\gamma}}) \not\subseteq H(K_{\boldsymbol{\gamma}}^{a})$ and $H(K_{\boldsymbol{\gamma}}^{a}) \not\subseteq H(K_{\boldsymbol{\gamma}})$ [Hefter & Ritter '14].

"Embedding Triple"

In general, $H(K_{\boldsymbol{\gamma}}) \not\subseteq H(K^a_{\boldsymbol{\gamma}})$ and $H(K^a_{\boldsymbol{\gamma}}) \not\subseteq H(K_{\boldsymbol{\gamma}})$ [Hefter & Ritter '14].

Remedy: Manipulate the Weights!

For $c > 0$ let $c\boldsymbol{\gamma} := (c\gamma_j)_{j\in\mathbb{N}}$. Note that $\mathrm{dec}(c\boldsymbol{\gamma}) = \mathrm{dec}(\boldsymbol{\gamma})$.

"Embedding Triple"

In general, $H(K_{\boldsymbol{\gamma}}) \not\subseteq H(K^a_{\boldsymbol{\gamma}})$ and $H(K^a_{\boldsymbol{\gamma}}) \not\subseteq H(K_{\boldsymbol{\gamma}})$ [Hefter & Ritter '14].

Remedy: Manipulate the Weights!

For $c > 0$ let $c\boldsymbol{\gamma} := (c\gamma_j)_{j\in\mathbb{N}}$. Note that $\mathrm{dec}(c\boldsymbol{\gamma}) = \mathrm{dec}(\boldsymbol{\gamma})$.

Proposition [G., Hefter, Hinrichs & Ritter'17]. *There exists* $0 < c_0 < 1$ *such that*

$$H(K^a_{c_0\boldsymbol{\gamma}}) \hookrightarrow H(K_{\boldsymbol{\gamma}}) \hookrightarrow H(K^a_{c_0^{-1}\boldsymbol{\gamma}}).$$

In general, $H(K^a_{c_0\boldsymbol{\gamma}}) \subsetneq H(K_{\boldsymbol{\gamma}}) \subsetneq H(K^a_{c_0^{-1}\boldsymbol{\gamma}})$.

Structure IV: Spaces of Increasing Smoothness

Let $(e_\nu)_{\nu\in\mathbb{N}_0}$ ONB of $L^2(D,\mu)$ with $e_0 = 1$.

Consider "Fourier weights" $\alpha_{\nu,j} \geq 1\,,\ \nu, j \in \mathbb{N}$, with

(A 1) $\alpha_{\nu,j} \leq \alpha_{\nu+1,j}$ and $\alpha_{\nu,j} \leq \alpha_{\nu,j+1}$ for all $\nu, j \in \mathbb{N}$,

(A 2) $\sum_{j\in\mathbb{N}} \sum_{\nu\in\mathbb{N}} \alpha_{\nu,j}^{-1} < \infty$.

Structure IV: Spaces of Increasing Smoothness

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Consider "Fourier weights" $\alpha_{\nu,j} \geq 1\,,\ \nu, j \in \mathbb{N}$, with

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(A 2) $\sum_{j \in \mathbb{N}} \sum_{\nu \in \mathbb{N}} \alpha_{\nu,j}^{-1} < \infty$.

For each variable $j \in \mathbb{N}$ define RK

$$k_j(x,y) = \sum_{\nu \in \mathbb{N}} \alpha_{\nu,j}^{-1} \cdot e_\nu(x) \cdot e_\nu(y), \qquad x, y \in \mathbb{R};$$

Then

$$H(1+k_j) = \Big\{ f \in L^2(D,\mu) \,:\, \sum_{\nu \in \mathbb{N}} \alpha_{\nu,j} \, \langle f, e_\nu \rangle_{L^2}^2 < \infty \Big\}.$$

Structure IV: Spaces of Increasing Smoothness

Let $(e_\nu)_{\nu\in\mathbb{N}_0}$ ONB of $L^2(D,\mu)$ with $e_0 = 1$.

Consider "Fourier weights" $\alpha_{\nu,j} \geq 1\,,\ \nu, j \in \mathbb{N}$, with

(A 1) $\alpha_{\nu,j} \leq \alpha_{\nu+1,j}$ and $\alpha_{\nu,j} \leq \alpha_{\nu,j+1}$ for all $\nu, j \in \mathbb{N}$,

(A 2) $\sum_{j\in\mathbb{N}} \sum_{\nu\in\mathbb{N}} \alpha_{\nu,j}^{-1} < \infty$.

For each variable $j \in \mathbb{N}$ define RK

$$k_j(x,y) = \sum_{\nu\in\mathbb{N}} \alpha_{\nu,j}^{-1} \cdot e_\nu(x) \cdot e_\nu(y), \qquad x, y \in \mathbb{R};$$

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Examples: Korobov spaces, Walsh spaces, Haar spaces, Hermite spaces,....

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Build "increasing smoothness RK"

$$K(\mathbf{x},\mathbf{y}) := \prod_{j\in\mathbb{N}} (1 + k_j(x_j, y_j)), \qquad \mathbf{x}, \mathbf{y} \in D^{\mathbb{N}}.$$

Result 3: Spaces of Increasing Smoothness

Define weights

$$\gamma_j := \sup_{\nu \in \mathbb{N}} \frac{\alpha_{\nu,1}}{\alpha_{\nu,j}}, \qquad j \in \mathbb{N}.$$

Interpretation of weights: $\sqrt{\gamma_j}$ norm of embedding $H(k_j) \hookrightarrow H(k_1)$.

Sequence $\boldsymbol{\gamma} = (\gamma_j)_{j \in \mathbb{N}}$ monotone decreasing. Assume $\sum_{j \in \mathbb{N}} \gamma_j < \infty$.

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Theorem 3 [G., Hefter, Hinrichs, Ritter & Wasilkowski'19;
G., Hinrichs, Ritter & Rüßmann'23+].

$$\min\left(\operatorname{dec}(1+k_1)\,,\ \frac{\operatorname{dec}(\boldsymbol{\gamma})-1}{2}\right) \leq \operatorname{dec}(K)$$
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Examples: Let $1 < r_1 \leq r_2 \leq r_3 \leq \dots$

Fourier weights of ...

- ... polynomial growth (PG): $\alpha_{\nu,j} := (\nu+1)^{r_j}$,
- ... exponential growth (EG): $\alpha_{\nu,j} := 2^{r_j \cdot \nu}$.

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For both cases (PG) & (EG): $\alpha_{1,j}^{-1} = 2^{-r_j}$ and $\gamma_j = 2^{r_1 - r_j}$, hence

$$\mathrm{dec}(\boldsymbol{\gamma}) = \mathrm{dec}((\alpha_{1,j}^{-1})_{j\in\mathbb{N}}) = \liminf_{j\to\infty}\left(\frac{\ln(2)}{\ln(j)}\, r_j\right),$$

resulting in matching bounds in Theorem 3.

L^2-Approximation

Solution Operator: Embedding operator $S : H(K) \to L^2(D^{\mathbb{N}}, \mu^{\mathbb{N}})$, $f \mapsto f$.

Algorithms: Linear algorithms $A : H(K) \to \mathbb{R}$ of form

$$A(f) := \sum_{i=1}^{m} w_i \cdot f\big(\mathbf{x}^{(i)}\big) \tag{3}$$

where $m \in \mathbb{N}$, $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)} \in D^{\mathbb{N}}$, and $w_1, \ldots, w_m \in L^2(D^{\mathbb{N}}, \mu^{\mathbb{N}})$.

Cost: We only account for function evaluations, fix nominal value $a \in \mathbb{R}$, ...

Error Criterion: Worst case randomized/deterministic error of algorithm A ...

Results: Theorem 1, 2, and 3 hold verbatim[1]! Again, upper error bounds are established via MDM based on Smolyak algorithms.

[1] But the univariate convergence rates of n-th minimal errors of integration and L^2-approximation may differ.

Summary

- Infinite-variate Integration:
  1. On spaces of integrands we first imposed structure (reproducing kernels, tensor product structure, weights, anchored function decomposition).
  2. Afterwards we fixed the setting (admissable algorithms, costs, and error criterion). In that setting...
  3. ... MDM achieved optimal results on anchored weighted RKHSs by exploiting the anchored function decomposition.
  4. ... an Embedding framework was used to prove that MDM achieve also optimal results on general weighted RKHSs.
  5. ... another embedding strategy was used to prove that MDM achieve also optimal results on RKHSs of increasing smoothness.
- Infinite-variate L^2-Approximation:
  Same holds for L^2-approximation (main results Theorem 1, 2 & 3 for integration hold verbatim for L^2-approximation).
- Concluding Remark: Our embedding techniques allow to transfer error bounds for linear algorithms for multi- or ∞-variate linear problems on one RKHS of weighted type or of increasing smoothness to different ones.

Sketch of Proof of Theorem 3

Consider *new* weighted RKs

$$k_j^{\mathrm{up}}(x,y) := \gamma_j\, k_1(x,y), \quad \text{and} \quad k_j^{\mathrm{low}}(x,y) := \alpha_{1,j}^{-1}\, e_1(x)e_1(y),$$

and put

$$K_{\boldsymbol{\gamma}}^{\mathrm{up}}(\mathbf{x},\mathbf{y}) := \prod_{j=1}^{\infty}(1+k_j^{\mathrm{up}}(x_j,y_j)), \quad \text{and} \quad K_{\alpha_{1,j}^{-1}}^{\mathrm{low}}(\mathbf{x},\mathbf{y}) := \prod_{j=1}^{\infty}(1+k_j^{\mathrm{low}}(x_j,y_j)).$$

Then

$$H(K_{\alpha_{1,j}^{-1}}^{\mathrm{low}}) \hookrightarrow H(K) \hookrightarrow H(K_{\boldsymbol{\gamma}}^{\mathrm{up}}).$$

Since $\mathrm{dec}(1+k_1^{\mathrm{up}}) = \mathrm{dec}(1+k_1)$ and $\mathrm{dec}(1+k_1) \geq \mathrm{dec}(K)$, we obtain

$$\begin{aligned}\min\left(\mathrm{dec}(1+k_1)\,,\ \frac{\mathrm{dec}(\boldsymbol{\gamma})-1}{2}\right) &\leq \mathrm{dec}(K) \\ &\leq \min\left(\mathrm{dec}(1+k_1)\,,\ \frac{\mathrm{dec}((\alpha_{1,j}^{-1})_{j\in\mathbb{N}})-1}{2}\right)\end{aligned}$$

Observation: $\dim(H(1+k_j^{\mathrm{low}})) = 2$, while $\dim(H(1+k_j)) = \infty$ and typically $H(1+k_{j+1}) \hookrightarrow H(1+k_j)$ compactly, while $H(1+k_{j+1}^{\mathrm{up}}) = H(1+k_j^{\mathrm{up}})$ with equivalent norms.

An Alternative Cost Model

Randomized Quadrature: $Q_n(f) = \sum_{i=1}^{n} w_i f(\mathbf{x}^{(i)})$, where $w_i \in \mathbb{R}$, $\mathbf{x}^{(i)} \in D^{v_i}$.

Unrestricted Subspace Sampling [Kuo, Sloan, Wasilkowski, Woźniakowski'10a]

$$\mathrm{cost}_{\mathrm{unr}}(Q_n) := E\left[\sum_{i=1}^{n} |\{j \in \mathbb{N} : x_j^{(i)} \neq a\}|\right]$$

Nested Subspace Sampling [Creutzig, Dereich, Müller-Gronbach, Ritter'09]

$$\mathrm{cost}_{\mathrm{nest}}(Q_n) := E\left[\sum_{i=1}^{n} \max\{j \in \mathbb{N} : x_j^{(i)} \neq a\}\right]$$

"Rule of Thumb": For USS use multivariate decomposition method (MDM), but for NSS use multilevel algorithms!

General references for multilevel algorithms: Heinrich '98, '01 (integral equations/parametric integration); Giles '08a, '08b; Giles, Higham & Mao (SDEs); Giles & Waterhouse '09 (QMC-ML); ... *now better look at* http://people.maths.ox.ac.uk/gilesm/mlmc_community.html

Multilevel for ∞-dimensional integration problem: Hickernell, Müller-Gronbach, Niu & Ritter '10; Niu, Hickernell, Müller-Gronbach & Ritter '11; G. '12; Baldeaux '12; G.'13; Baldeaux & G. '14; Dick & G. '14a; Hefter '14; G., Hefter, Hinrichs & Ritter '17.

Comparison: Multilevel Algorithms and MDM

QMC-Multilevel Algorithm: $Q_L^{\mathrm{ML}}(f) := \sum_{\ell=1}^{L} Q_\ell(f)$, where

$$Q_\ell(f) := \frac{1}{n_\ell} \sum_{i=1}^{n_\ell} \Big(f(x_1^{(\ell,i)}, \ldots, x_{2^\ell}^{(\ell,i)}, a, a, \ldots) - f(x_1^{(\ell,i)}, \ldots, x_{2^{\ell-1}}^{(\ell,i)}, a, a, \ldots) \Big)$$

Let $f \in H(K_\gamma^a)$ with anchored function decomposition $f = \sum_{u \subset_f \mathbb{N}} f_u$.

Solution operator:

$$S(f) = \int_{D^{\mathbb{N}}} f(\mathbf{x}) \, \mathrm{d}\mu^{\mathbb{N}}(\mathbf{x}) = \sum_{u \subset_f \mathbb{N}} \int_{D^u} f_u(\mathbf{x}_u) \, \mathrm{d}\mu^u(\mathbf{x}_u)$$

Multivariate decomposition method:

$$Q^{\mathrm{MDM}}(f) = \sum_{u \subset_f \mathbb{N}} Q_{u,n_u}(f_u)$$

Multilevel algorithm:

$$Q^{\mathrm{ML}}(f) = \sum_{\ell=1}^{L} \Bigg(\sum_{\substack{u \subseteq \{1,2,\ldots,2^\ell\} \\ u \not\subseteq \{1,2,\ldots,2^{\ell-1}\}}} Q_\ell(f_u) \Bigg)$$

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Cost:
$$\mathrm{cost}_{\mathrm{nest}}(Q_L^{\mathrm{ML}}) = \mathrm{cost}_{\mathrm{unr}}(Q_L^{\mathrm{ML}}) \le \sum_{\ell=1}^{L} 2\,n_\ell\,2^\ell$$

Multivariate decomposition method: Q^{MDM}:

$$Q^{\mathrm{MDM}}(f) = \sum_{u\in\mathcal{A}_\varepsilon} Q_{u,n_u}(f_u)$$

$f_u(\mathbf{x}) = \sum_{v\subseteq u}(-1)^{|u\setminus v|} f(\mathbf{x}_v)$

Cost:
$$\mathrm{cost}_{\mathrm{unr}}(Q_L^{\mathrm{MDM}}) = O\left(\sum_{u\in\mathcal{A}_\varepsilon} n_u 2^{|u|}|u|\right)$$

(Recall: $f_u(\mathbf{x}) = \sum_{v\subseteq u}(-1)^{|u\setminus v|} f(\mathbf{x}_v)$.
Cost for evaluating f_u in unrestricted model: $O(2^{|u|}|u|)$.
"Typically": $u\in\mathcal{A}_\varepsilon \implies |u| = o(\ln(1/\varepsilon))$.)

Example of Non-Anchored Decomposition: ANOVA Decomp.

ANOVA condition $\int_D k(x,y)\, d\mu(y) = 0 \ \ \forall x \in D$ implies that function decomp. $f = \sum_{u \subset_f \mathbb{N}} f_u$ in $H(K_{\boldsymbol{\gamma}})$ is ANOVA decomp. (in $L^2(D^{\mathbb{N}}, \mu^{\mathbb{N}})$):

$$f_{\emptyset}(\mathbf{x}) := \int_{D^{\mathbb{N}}} f(\mathbf{y})\, \mathrm{d}\mu^{\mathbb{N}}(\mathbf{y})\,,$$
$$f_u(\mathbf{x}) := \int_{D^{\mathbb{N}\setminus u}} f(\mathbf{x}_u, \mathbf{y}_{\mathbb{N}\setminus u})\, \mathrm{d}\mu^{\mathbb{N}\setminus u}(\mathbf{y}_{\mathbb{N}\setminus u}) - \sum_{v \subsetneq u} f_v(\mathbf{x})\,,$$

where $\mathbf{x}_u := (x_j)_{j\in u}$, $\mathbf{y}_{\mathbb{N}\setminus u} := (y_j)_{j \in \mathbb{N}\setminus u}$.

This implies

$$\|f\|^2_{L^2(D^{\mathbb{N}}, \mu^{\mathbb{N}})} = \sum_{u \subset_f \mathbb{N}} \|f_u\|^2_{L^2(D^u, \mu^u)} \quad \text{and} \quad \mathrm{Var}(f) = \sum_{u \subset_f \mathbb{N}} \mathrm{Var}(f_u).$$

(∞-variate ANOVA decomposition was studied in:
Hickernell, Müller-Gronbach, Niu, Ritter '10; G., Baldeaux '14; Dick, G. '14b; Griebel, Kuo, Sloan '16;...)

Some Motivation

Given: $X = (X_t)_{t\in[0,1]}$ stochastic process with series expansion

$$X_t = \sum_{j=1}^{\infty} \xi_j e_j(t),$$

$e_1, e_2, \ldots$ deterministic functions on $[0,1]$, $\xi_1, \xi_2, \ldots$ i.i.d. r.v. with distribution μ on D and $E[\xi_i] = 0$.

Examples: Consider $D = \mathbb{R}$ and $\mu = N(0,1)$.

- Karhunen-Loéve expansion of zero-mean Gaussian process X; then $(e_j)_{j\in\mathbb{N}}$ orthogonal system in $L^2([0,1])$ with $\sum_{j=1}^{\infty} \|e_j\|^2_{L^2([0,1])} < \infty$.
- Lévy-Ciesielski or Brownian Bridge construction of Brownian motion X; then $e_1, e_2, \ldots$ Schauder functions.

Task: For given functional φ on path space of X evaluate $E[\varphi(X)]$.

Observation:

$$E[\varphi(X)] = \int_{D^{\mathbb{N}}} f(\mathbf{x}) \, \mathrm{d}\mu^{\mathbb{N}}(\mathbf{x}), \quad \text{where} \quad f(\mathbf{x}) = \varphi\Big(\sum_{j=1}^{\infty} x_j e_j\Big).$$

Computational Approach

Step 1: "Truncation":

$$\sum_{j=1}^{\infty} \xi_j e_j \approx \sum_{j=1}^{d} \xi_j e_j \quad \text{and} \quad E[\varphi(X)] \approx E\left[\varphi\left(\sum_{j=1}^{d} \xi_j e_j\right)\right]$$

Step 2: Approximate

$$E\left[\varphi\left(\sum_{j=1}^{d} \xi_j e_j\right)\right] = \int_{D^{\mathbb{N}}} f(x_1, \ldots, x_d, 0, 0, \ldots)\, \mathrm{d}\mu^{\mathbb{N}}(\mathbf{x})$$
$$= \int_{D^d} f(x_1, \ldots, x_d, 0, 0, \ldots)\, \mathrm{d}\mu^{d}(x_1, \ldots, x_d)$$

with algorithms for multivariate integration (recall $f(\mathbf{x}) = \varphi(\sum_{j=1}^{\infty} x_j e_j)$).