Adaptive step-size control for global approximation of SDEs driven by countably dimensional Wiener process

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Main references

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Preliminaries - our model

- $(\Omega, \Sigma, (\Sigma_t)_{t\geq 0}, \mathbb{P})$ – probability space with sufficiently rich filtration,
- $T > 0$ – termination time,
- $W = [W_1, W_2, \ldots]^T$ – countably dimensional Wiener process,
- $x_0 \in \mathbb{R}$ - initial value.

Let us consider the following one-dimensional SDE

$$X(t) = x_0 + \int_0^t a(s, X(s))ds + \sum_{j=1}^{+\infty} \int_0^t \sigma_j(s)dW_j(s), \ t \in [0, T], \tag{1}$$

where $\sigma(s) = (\sigma_1(s), \sigma_2(s), \ldots) \in \ell^2(\mathbb{R})$ for each $s \in [0, T]$.

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where $\sigma(s) = (\sigma_1(s), \sigma_2(s), \ldots) \in \ell^2(\mathbb{R})$ for each $s \in [0, T]$.

Preliminaries - cont'd

$\|\cdot\|_{\ell^2}$ - $\ell^2(\mathbb{R})$ norm;

$\|\cdot\|_2$ - L^2 norm on $\Omega \times [0, T]$.

Assumption (drift)

(A) We assume that the drift coefficient $a : [0, T] \times \mathbb{R} \mapsto \mathbb{R}$ *belongs to* $\mathcal{C}^{1,2}([0, T] \times \mathbb{R})$ *and satisfies the following conditions:*

(A1) $|a(t, x) - a(s, x)| \leq C_1(1 + |x|)|t - s|$ *for all* $t, s \in [0, T],\ x \in \mathbb{R}$,

(A2) $|a(t, 0)| \leq C_1$ *for all* $t \in [0, T]$,

(A3) $|a(t, x) - a(t, y)| \leq C_1|x - y|$ *for all* $x, y \in \mathbb{R},\ t \in [0, T]$,

(A4) $\left|\frac{\partial a}{\partial x}(t, x) - \frac{\partial a}{\partial x}(t, y)\right| \leq C_1|x - y|$ *for all pairs* $(t, x), (t, y) \in [0, T] \times \mathbb{R}$

for some $C_1 > 0$.

Preliminaries - cont'd

Let $\delta = (\delta(k))_{k=1}^{\infty} \subset \mathbb{R}$ be a positive, strictly decreasing sequence vanishing at infinity.

By $\mathcal{G}_\delta$ we denote a set of all non-decreasing sequences $G = (G(n))_{n=1}^{\infty} \subset \mathbb{N}$ such that $G(n) \to +\infty$ and

$$\lim_{n \to +\infty} n^{1/2}\,\delta\big(G(n)\big) = 0. \tag{2}$$

The set of sequences G falling under (2) is non-empty.

Preliminaries - cont'd

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The set of sequences G falling under (2) is non-empty.

Preliminaries - cont'd

Assumption (diffusion)

(S) We assume that diffusion coefficient $\sigma = (\sigma_1, \sigma_2, \dots) : [0, T] \mapsto \ell^2(\mathbb{R})$ *satisfies the following conditions:*

(S1) $\|\sigma(0)\|_{\ell^2} \le C_2$,

(S2) $\|\sigma(t) - \sigma(s)\|_{\ell^2} \le C_2 |t - s|$ *for all* $t, s \in [0, T]$,

(S3) $\|\sigma(t) - P_k\sigma(t)\|_{\ell^2} \le C_2\,\delta(k)$ *for all* $k \in \mathbb{N}, t \in [0, T]$.

where $C_2 > 0$, *and* $\delta = (\delta(k))_{k=1}^{+\infty}$ *is as before.*

Here we leverage the notation $P_k : \ell^2(\mathbb{R}) \mapsto \ell^2(\mathbb{R})$, where $P_k x = (x_1, x_2, \dots, x_k, 0, 0, \dots)$. We denote $\sigma^k(t) = (P_k\sigma)(t)$ and $P_\infty = Id$.

Truncated dimension approximation

Let $M \in \mathbb{N}$. Our idea is to provide the approximation of 'truncated' solution $X^M = (X^M(t))_{t\in[0,T]}$, defined as follows

$$X^M(t) = x_0 + \int_0^t a(s, X^M(s))\,\mathrm{d}s + \int_0^t \sigma^M(s)\,\mathrm{d}W(s), \quad t \in [0, T], \tag{3}$$

where

$$\int_0^t \sigma^M(s)\,\mathrm{d}W(s) = \sum_{j=1}^{M} \int_0^t \sigma_j(s)\,\mathrm{d}W_j(s).$$

Truncated dimension approximation-cont'd

Proposition 1.

For every $M \in \mathbb{N} \cup \{\infty\}$ the equation (3) admits a unique strong solution $X = (X(t))_{t \in [0,T]}$. Moreover, there exists $K_1 \in (0, +\infty)$, such that for every $M \in \mathbb{N} \cup \{\infty\}$ we have that

$$\mathbb{E}\Big(\sup_{0 \leq t \leq T} |X^M(t)|^2\Big) \leq K_1.$$

Proposition 2.

There exists $K_1 \in (0, +\infty)$ such that for any $M \in \mathbb{N}$ it holds

$$\sup_{0 \leq t \leq T} \|X(a, \sigma, x_0)(t) - X^M(a, \sigma, x_0)(t)\|_{L^2(\Omega)} \leq K_1 \delta(M).$$

Minimal error bounds - definition of an admissible method

$\overline{X} = (\overline{X}_{M_n,\bar{k}_n})_{n=1}^{\infty}$ is defined by $\overline{X} = (\bar{M}, \bar{\Delta}, \bar{\mathcal{N}}, \bar{\phi})$, where:

- $\bar{M} = (M_n)_{n=1}^{\infty} \in \mathcal{G}_\delta$.
- $(\bar{\Delta}_n)_{n=1}^{\infty}$ is a sequence of (possibly) non-expanding partitions of the interval $[0, T]$

  $$\bar{\Delta}_n : \quad 0 = \bar{t}_{0,n} < \bar{t}_{1,n} < \ldots < \bar{t}_{\bar{k}_n-1,n} < \bar{t}_{\bar{k}_n,n} = T,$$

  where for some $\overline{C}_1, \overline{C}_2 > 0$ it holds

  $$\overline{C}_1\, n \geq \bar{k}_n \geq \overline{C}_2\, n^{1/2}, \quad n \geq n_0(\overline{X}) \in \mathbb{N}.$$

- $\bar{\mathcal{N}} = (\mathcal{N}_{M_n,\bar{k}_n}(W))_{n=1}^{\infty}$ is a sequence of information vectors

  $$\mathcal{N}_{M_n,\bar{k}_n}(W) = \bigoplus_{k=1}^{M_n} \left[W_k(\bar{t}_{1,n}), W_k(\bar{t}_{2,n}), \ldots, W_k(\bar{t}_{\bar{k}_n,n})\right].$$

- $\bar{\phi} = (\phi_n)_{n=1}^{\infty}$ is a sequence of Borel mappings $\phi_n : \mathbb{R}^{n \cdot M_n} \mapsto L^2([0, T])$, such that

  $$\phi_n(\mathcal{N}_{M_n,\bar{k}_n}(W)) = \overline{X}_{M_n,\bar{k}_n}, \quad n \in \mathbb{N}.$$

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Minimal error bounds - admissible algorithms

The class of algorithms satisfying above conditions is denoted by χ_{noneq}. We also distinguish a subclass $\chi_{eq} \subset \chi_{noneq}$ of methods leveraging equidistant partitions

$$\chi_{eq} = \Big\{ \overline{X} \in \chi_{noneq} \,|\, \exists_{n_0 = n_0(\overline{X})} : \forall_{n \geq n_0} \; \bar{\Delta}_n = \{ jT/n \; : \; j = 0, 1, \ldots, \bar{k}_n \} \Big\}.$$

Similarly, for a fixed truncation sequence $\bar{M}$, we define corresponding subclasses $\chi^{\bar{M}}_{noneq}$ and $\chi^{\bar{M}}_{eq}$.

Minimal error bounds - cont'd

Informational cost of the algorithm $\overline{X}_{M_n,\bar{k}_n}$:

$$\mathrm{cost}(\overline{X}_{M_n,\bar{k}_n}) = \begin{cases} M_n \cdot \bar{k}_n, & \text{when } \sigma \not\equiv 0, \\ 0, & \text{when } \sigma \equiv 0. \end{cases}$$

Global approximation error for $\overline{X}_{M_n,\bar{k}_n}$:

$$\|X - \overline{X}_{M_n,\bar{k}_n}\|_2 = \left(\mathbb{E}\int_0^T |X(t) - \overline{X}_{M_n,\bar{k}_n}(t)|^2 \,\mathrm{d}t\right)^{1/2}, \quad n \in \mathbb{N}.$$

Minimal error bounds - cont'd

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Minimal error bounds - notation

Let $(a(n))_{n=1}^{\infty}, (b(n))_{n=1}^{\infty}$ be two sequences of positive numbers.

$$a(n) \approx b(n) \quad :\Leftrightarrow \quad \lim_{n \to +\infty} \frac{a(n)}{b(n)} = 1.$$

Furthermore, we will say that

$$a(n) \lessapprox b(n) \quad :\Leftrightarrow \quad \limsup_{n \to +\infty} \frac{a(n)}{b(n)} \leq 1.$$

Minimal error bounds

Theorem 2 (S., 2023), part 1.

Let $\bar{M} = (M_n)_{n=1}^{\infty} \in \mathcal{G}_\delta$. We have the following asymptotic bound

$$\inf_{\overline{X} \in \chi^{\bar{M}}_{noneq}} \left(\mathrm{cost}(\overline{X}_{M_n,\bar{k}_n})\right)^{1/2} \|\overline{X}_{M_n,\bar{k}_n} - X\|_2 \gtrapprox \frac{M_n^{1/2}}{\sqrt{6}} \int_0^T \|\sigma(t)\|_{\ell^2} \mathrm{d}t.$$

Theorem 2 (S., 2023), part 2.

Let $\bar{M} = (M_n)_{n=1}^{\infty} \in \mathcal{G}_\delta$. We have the following asymptotic bound

$$\inf_{\overline{X} \in \chi^{\bar{M}}_{eq}} \left(\mathrm{cost}(\overline{X}_{M_n,\bar{k}_n})\right)^{1/2} \|\overline{X}_{M_n,\bar{k}_n} - X\|_2 \gtrapprox M_n^{1/2} \sqrt{\frac{T}{6}} \left(\int_0^T \|\sigma(t)\|^2_{\ell^2}\,\mathrm{d}t\right)^{1/2}.$$

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Theorem 2 (S., 2023), part 2.

Let $\bar{M} = (M_n)_{n=1}^{\infty} \in \mathcal{G}_\delta$. We have the following asymptotic bound

$$\inf_{\overline{X} \in \chi^{\bar{M}}_{eq}} \left(\mathrm{cost}(\overline{X}_{M_n,\bar{k}_n}) \right)^{1/2} \|\overline{X}_{M_n,\bar{k}_n} - X\|_2 \gtrapprox M_n^{1/2} \sqrt{\frac{T}{6}} \left(\int_0^T \|\sigma(t)\|_{\ell^2}^2 \, \mathrm{d}t \right)^{1/2}.$$

Minimal error bounds -cont'd

Let us denote by

$$\mathcal{C}_{noneq} = \frac{1}{\sqrt{6}} \int_0^T \|\sigma(t)\|_{\ell^2} \, \mathrm{d}t$$

and

$$\mathcal{C}_{eq} = \sqrt{\frac{T}{6}} \left(\int\limits_0^T \|\sigma(t)\|_{\ell^2}^2 \, \mathrm{d}t \right)^{1/2}$$

the constants appearing on the RHS in Theorem 2, respectively.

Investigation of an optimal algorithm in $\chi_{eq}^{\bar{M}}$

For fixed truncation level sequence $\bar{M} = (M_n^*)_{n=1}^{\infty} \in \mathcal{G}_\delta$ and $n \in \mathbb{N}$, we define truncated dimension Euler scheme $X_{M_n^*,n}^{Eq*}$ based on the equidistant partitions $\Delta_n^{eq} = (t_{j,n}^{eq})_{j=0}^{n}$

$$\begin{cases} X_{M_n^*,n}^{Eq*}(0) = x_0 \\ X_{M_n^*,n}^{Eq*}(t_{j+1,n}^{eq}) = X_{M_n^*,n}^{Eq*}(t_{j,n}^{eq}) + a\big(t_{j,n}^{eq}, X_{M_n^*,n}^{Eq*}(t_{j,n}^{eq})\big)(t_{j+1,n}^{eq} - t_{j,n}^{eq}) \\ \qquad\qquad\qquad\qquad + \sigma^{M_n^*}(t_{j,n}^{eq})\big(W(t_{j+1,n}^{eq}) - W(t_{j,n}^{eq})\big), \\ \qquad j = 0, 1, \ldots, n-1. \end{cases}$$

The associated process $(X_{M_n^*,n}^{Eq*}(t))_{t\in[0,T]}$ is obtained by linear interpolation.

Investigation of optimal algorithm in $\chi_{eq}^{\bar{M}}$ - cont'd

We show that

$$n^{1/2}\|X - X^{Eq*}_{M_n^*,n}\|_2 \approx \left(\frac{T}{6}\sum_{j=0}^{n-1}\|\sigma^{M_n^*}(t^{eq}_{j,n})\|^2_{\ell^2}\,\frac{T}{n}\right)^{1/2}, \quad n \to +\infty.$$

As a result,

$$(M_n^* n)^{1/2}\|X - X^{Eq*}_{M_n^*,n}\|_2 \approx (M_n^*)^{1/2}\sqrt{\frac{T}{6}}\left(\int_0^T \|\sigma(t)\|^2_{\ell^2}\,\mathrm{d}t\right)^{1/2}, n \to +\infty.$$

Algorithm with adaptive step-size control

Let us fix $n \in \mathbb{N}$ and $\bar{M} = (M_n^*)_{n=1}^{\infty} \in \mathcal{G}_\delta$. The proposed scheme $X^{step}_{M_n^*,k_n^*}$ uses the following adaptive path-independent step-size control: $\hat{t}_{0,n} := 0$ and

$$\hat{t}_{j+1,n} := \hat{t}_{j,n} + \frac{T}{n \max\{\varepsilon_n, \|\sigma^{M_n^*}(\hat{t}_{j,n})\|_{\ell^2}\}}, \quad j = 0, 1, \ldots, k_n^* - 1,$$

where $k_n^* = \inf\{j \in \mathbb{N} \mid \hat{t}_{j,n} \geq T\}$, and $\bar{\varepsilon} = (\varepsilon_n)_{n=1}^{\infty} \subset \mathbb{R}_+$ is a non-increasing sequence satisfying

$$\lim_{n\to+\infty} \varepsilon_n = \lim_{n\to+\infty} \frac{1}{n\,\varepsilon_n^2} = 0.$$

Algorithm with adaptive step-size control - cont'd

Now we set

$$\begin{cases} X^{step}_{M^*_n,k^*_n}(0) &= x_0 \\ X^{step}_{M^*_n,k^*_n}(\hat{t}_{j+1,n}) &= X^{step}_{M^*_n,k^*_n}(\hat{t}_{j,n}) + a(\hat{t}_{j,n}, X^{step}_{M^*_n,k^*_n}(\hat{t}_{j,n}))(\hat{t}_{j+1,n} - \hat{t}_{j,n}) \\ & \qquad + \sigma^{M^*_n}(\hat{t}_{j,n})(W(\hat{t}_{j+1,n}) - W(\hat{t}_{j,n})), \\ & j = 0, 1, \ldots, k^*_n - 1, \end{cases}$$

with $\hat{t}_{k^*_n,n}$ replaced with T.

Algorithm with adaptive step-size - cont'd

We introduce the partition Δ_n^* by taking $t_{j,n}^* = \hat{t}_{j,n},\ j = 0, 1, \dots, k_n^* - 1$, and $t_{k_n^*,n}^* = T$.

Ultimately, for each j, we perform linear interpolation between $X_{M_n^*,k_n^*}^{step}(t_{j,n}^*)$ and $X_{M_n^*,k_n^*}^{step}(t_{j+1,n}^*)$ to obtain the final process

$$X_{M_n^*,k_n^*}^* = (X_{M_n^*,k_n^*}^{step}(t))_{t\in[0,T]}.$$

Properties

Under the assumptions (A1) - (A4), (S1)-(S3):

a) The proposed method $X_{M_n^*,k_n^*}^{step}$ with step-size control is an element of χ_{noneq} and attains point T.

b) k_n^* is deterministic and $\lim_{n\to+\infty} k_n^*(\sigma) = +\infty$;

c) $\max_{0\le j\le k_n^*-1} (t_{j+1,n}^* - t_{j,n}^*) \le \frac{T}{n\varepsilon_n} \to 0, \quad n \to +\infty.$

Algorithm with adaptive step-size - cont'd

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b) k_n^* is deterministic and $\lim\limits_{n\to+\infty} k_n^*(\sigma) = +\infty$;

c) $\max\limits_{0\leq j\leq k_n^*-1}(t_{j+1,n}^* - t_{j,n}^*) \leq \frac{T}{n\varepsilon_n} \to 0, \quad n \to +\infty.$

Recall that by $\mathcal{G}_\delta$ we denote a set of all non-decreasing sequences $G = (G(n))_{n=1}^{\infty} \subset \mathbb{N}$ such that $G(n) \to +\infty$ and

$$\lim_{n\to+\infty} n^{1/2}\,\delta\big(G(n)\big) = 0.$$

Theorem 3. (S. 2023)

Let a, σ satisfy conditions (A) and (S) with sequence δ, respectively. Then, for every method $\bar{X} = (\overline{X}_{M_n,\bar{k}_n})_{n=1}^{\infty} \in \chi_\diamond, \diamond \in \{noneq, eq\}$, we have

$$\big(\mathrm{cost}(\overline{X}_{M_n,\bar{k}_n})\big)^{1/2}\,\big\|X - \overline{X}_{M_n,\bar{k}_n}\big\|_2 \gtrapprox \big(\delta^{-1}(n^{-1/2})\big)^{1/2}\,\mathcal{C}_\diamond, \quad n \to +\infty.$$

Asymptotically (almost) optimal algorithms for classes χ_{eq}, χ_{noneq} - cont'd

Theorem 3. (S. 2023) - part 2.

For every truncation level sequence M_n with $\delta^{-1}(n^{-1/2}) = o(M_n), n \to +\infty$, there exists a sequence $M^* = (M_n^*)_{n=1}^{\infty} \in \mathcal{G}_\delta$ such that $M_n^* = o(M_n),\ n \to +\infty$, and:

a) the truncated-dimension Euler algorithm with adaptive path-independent step-size control $X^* = \left(X^*_{M_n^*, k_n^*}\right)_{n=1}^{\infty} \in \chi_{noneq}$ satisfying

$$\left(\mathrm{cost}(\bar{X}_{M_n^*, k_n^*})\right)^{1/2} \left\|X - \bar{X}_{M_n^*, k_n^*}\right\|_2 \lessapprox \sqrt{\frac{M_n^*}{6}} \int_0^T \|\sigma(t)\|_{\ell^2}\, \mathrm{d}t, \quad n \to +\infty;$$

b) the truncated-dimension Euler algorithm $X^{Eq*} = \left(X^{Eq*}_{M_n^*, n}\right)_{n=1}^{\infty} \in \chi_{eq}$, based on the sequence of equidistant meshes, and satisfying

$$\left(\mathrm{cost}(X^{Eq*}_{M_n^*, n})\right)^{1/2} \left\|X - X^{Eq*}_{M_n^*, n}\right\|_2 \lessapprox \sqrt{\frac{M_n^* T}{6}} \left(\int_0^T \|\sigma(t)\|^2_{\ell^2}\, \mathrm{d}t\right)^{1/2}, \quad n \to +\infty.$$

Numerical experiments - chosen model

$$a(t,x) = (t+2)(x-1), \quad t \in [0,T],\ x \in \mathbb{R},$$

$$\sigma_k(t) = \frac{e^{2t}+2}{(k+1)^p \sqrt{\log(k+1)}} \quad t \in [0,T],\ k = 1,2,\ldots,$$

where $p > 1/2$.

For all $l \in \mathbb{N},\ l > 1$ it holds

$$\|\sigma(t) - P_l\sigma(t)\|_{\ell^2}^2 = \left| \sum_{k=l}^{+\infty} \frac{(e^{2t}+2)^2}{(k+1)^{2p}\log(k+1)} \right| \leq (e^{2T}+2)^2 \left| \int_{(2p-1)\log(l+1)}^{+\infty} e^{-v}\, v^{-1}\, \mathrm{d}v \right|,$$

and the integral appearing above is equal to the upper incomplete gamma function $\Gamma\Big(1, (2p-1)\log(l+1)\Big)$. Therefore,

$$\|\sigma(t) - P_l\sigma(t)\|_{\ell^2} \leq (e^{2T}+2)e^{-0.5(2p-1)\log(l+1)} = (e^{2T}+2)\Big(l+1\Big)^{1/2-p}, \quad l \in \mathbb{N}.$$

Numerical experiments - chosen model

$$\|\sigma(t) - P_n\sigma(t)\|_{\ell^2} \leq K(T)n^{1/2-p}, \quad n \in \mathbb{N}.$$

Therefore, we can assume

$$\delta(n) \approx n^{1/2-p} \quad \Rightarrow \quad \delta^{-1}(n^{-1/2}) \approx n^{\frac{1}{2p-1}}.$$

In our simulations, we set $x_0 = 0.9$, $T = 1.5$, $p = 0.9$, hence the admissible truncation levels can be of the form

$$\mathcal{G}_\delta \ni M_n \gtrapprox n^{5/4+\varepsilon}, n \to +\infty,$$

for some $\varepsilon > 0$. Our final choice is to take $M_n^* = 0.15 \cdot n^{1.28}$ and $\varepsilon_n = n^{-0.3}$, $n \in \mathbb{N}$.
We also note that $\mathcal{C}_{eq} = 4.550580...$, while $\mathcal{C}_{noneq} = 3.873137....$

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Numerical experiments - cont'd

Our target is to verify if the empirical ratio $\hat{\mathcal{C}}_{noneq}/\hat{\mathcal{C}}_{eq}$ matches the theoretical one (0.8511). To this end, we leverage the following metric

$$\mathrm{error}(X^{alg}) := \left(\frac{1}{K} \sum_{l=1}^{K} \mathcal{Q}\left(|X_l^{alg}(a, b, W^{(l)}) - X_{W_{ratio} M_n^*, n^*, l}(a, b, W^{(l)})|^2 \right) \right)^{1/2},$$

where:

- K is a number of simulated trajectories;
- X_l^{alg}, $X_{W_{ratio} M_n^*, n^*, l}$, and $W^{(l)}$ are the l–th generated trajectories of the corresponding processes $X^{alg} \in \{X^{Eq*}_{M_n^*, k_n^*}, X^{step}_{M_n*, k_n^*}\}$, $l = 1, \dots, K$;
- $\mathcal{Q}$ is a composite Simpson quadrature based on: the time points for which X^{alg} is evaluated, and the midpoints of the corresponding subintervals;
- rare-fine grid approach is utilised due to the fact that the exact solution formula contains stochastic integrals. The fine grid method is always based on $MAX\_n = 10^6$ equidistant nodes and uses $W_{ratio} \cdot M_n^*$ – dimensional Wiener process.

Numerical experiments - cont'd

n	k_n^*	M_n^*	K	W_{ratio}	Improvement ratio
1000	7832	1037	1000	2.0	0.977977
2000	15686	2520	1000	2.0	0.945800
5000	39249	8142	250	1.5	0.915620
10000	78520	19773	94	1.5	0.976337

Table: Simulation results for $X^{Eq*}_{M_n^*,n}$ and $X^{step}_{M_n^*,k_n^*}$.

Summary and conclusions

- We constructed truncated dimension Euler schemes: a) based on equidistant mesh; b) with adaptive step-size control;

  in a model with additive, countably dimensional structure of noise.
- We derived lower bound for the exact asymptotic error behaviour. Those estimates show that in order to decrease the minimal error, a significant additional cost needs to be taken.
- We proved that constructed methods are optimal / almost optimal in the considered (sub)classes.
- the conclusions from our experiments seem to be in line with the derived theoretical results.

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Future work

- Considering algorithms with additional adaptation with respect to the Wiener process coordinates,
- Extending present results linked to global approximation problem to the models with $\sigma = \sigma(t, x)$ and possibly jumps,
- Case study - efficient implementation of constructed algorithms by using GPU computational power.

Thank you for your attention!