Uniform in time approximations

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MCMC2023

Overview

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- General approach
- Several instances to which the approach is applicable
- Strengths and limitations

Context: UiT approximations

- Suppose X_t solves a (one-dimensional, elliptic) SDE

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  1. The dynamics to approximate
  2. The approximation
- I am not considering time-averages (though you get them as a consequence - modulo having an invariant measure)
- I don't necessarily want to approximate the invariant measure, I want to approximate the law at time t, for every $t > 0$

How do we go about this?

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- $u(t,x)$ solves a PDE

$$\partial_t u(t,x) = \mathcal{L}u(t,x) \quad u(0,x) = f(x)\,.$$

$$\mathcal{L} = b(x)\partial_x + \frac{1}{2}\sigma^2(x)\partial_{xx}$$

General Approach

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$$
\begin{aligned}
& \mathbb{E}f(X_t) - \mathbb{E}f(Y_t^\delta) \\
= \quad & (\mathcal{P}_t f)(x) - (\mathcal{P}_t^\delta f)(x) \\
= \quad & \int_0^t ds\, \mathcal{P}_{t-s}^\delta (\mathcal{L} - \mathcal{L}^\delta) \mathcal{P}_s f(x, y) \\
\leq \quad & \int_0^t \|(\mathcal{L} - \mathcal{L}^\delta)(\mathcal{P}_s f)\|_\infty ds \\
\simeq \quad & \delta \int_0^t \|\partial_x^2 (\mathcal{P}_s f)\|_\infty ds
\end{aligned}
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Key stability condition

$$\boxed{|\partial_x(\mathcal{P}_t f)(x)| \leq ce^{-\lambda t}} \quad \text{for some } \lambda > 0 \quad \text{(DE)}$$

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- Why is this a stability condition?

$$\int_0^\ell \frac{d}{du}(\mathcal{P}_t f)(\gamma(u))du = (\mathcal{P}_t f)(y) - (\mathcal{P}_t f)(x) = \int_0^\ell (\partial_x \mathcal{P}_t)(\gamma(u))du$$

Averaging

- Fast-slow dynamics, $(X_t, Y_t) \in \mathbb{R}^d \times \mathbb{R}^n$

$$\begin{aligned} dX_t &= f(X_t, Y_t)dt + g(X_t, Y_t)dW_t \\ dY_t &= \frac{1}{\epsilon}h(X_t, Y_t)dt + \frac{1}{\sqrt{\epsilon}}\sigma(X_t, Y_t)dB_t, \quad 0 < \epsilon \ll 1\,. \end{aligned}$$

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- Limiting dynamics

$$d\bar{X}_t = F(\bar{X}_t)dt + G(\bar{X}_t)dW_t$$

where

$$F(x) = \int f(x, y)\mu^x(dy), \; G(x) = \int g(x, y)\mu^x(dy)$$

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- Conceptual problems in practice if you don't have UiT approximation

- Associated Poisson equation:

$$(L^x f)(x, y) = \varphi(x, y)$$

- L^x is the generator of the fast process

$$dY_t = h(X_t, Y_t)dt + \sigma(X_t, Y_t)dB_t$$

namely

$$L^x f = h(x, y)\partial_y f + \sigma^2(x, y)\partial_{yy} f$$

$$L^x f = \varphi$$

- Representation formula

$$f(x, y) = \int_0^\infty (P_t^x \varphi)(y) dt$$

- smoothness of solution, in both x and y
- quantify how solution $f(x, y)$ varies as x varies - similarly for μ^x

Multiscale Problems for Interacting Particle Systems

- Particle system interacting with fast network

$$dX_t^i = -\nabla V(X_t^i)dt + \sum_{j\neq i} A_{ij}^{\varepsilon}(t)K(X_t^i - X_t^j)dt + \sqrt{2D}dB_t^i, \qquad i = 1...N$$

$$dA_{ij}^{\varepsilon}(t) = -A_{ij}^{\varepsilon}(t_-)dN_{ij}^{d,\varepsilon}(t) + [1 - A_{ij}^{\varepsilon}(t_-)]\mathbf{1}_{\{|X_t^i - X_t^j| \leq R\}}dN_{ij}^{f,\varepsilon}(t)$$

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- Main features:
  - large number of particles $N \to \infty$ and fast network evolution $\epsilon \to 0$
  - the network is *sparse*, i.e. the interaction is not of mean-field type

Are Derivative estimates hard to obtain?

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By now various simple criteria

- Numerical methods

[1] D.Crisan, P. Dobson, M. O. *Uniform in time estimates for the weak error of the Euler method*, Trans. AMS.

[2] L. Angeli, D. Crisan, M.O. *Uniform in time convergence of numerical schemes for SDEs via Strong Exponential Stability: Euler methods, Split-Step and Tamed Schemes.* arxiv

- Particle systems

[3] J. Barre, P. Dobson, M. O., E. Zatorska. *Non mean field fast network: uniform in time averaging*, SIAM Analysis

- Averaging

[4] B. Goddard, M. Ottobre, P. Dobson, I. Souttar *Poisson equations and uniform in time averaging* arxiv

- Analytic approach
  - S. Mischler, C. Mouhot. *Kac's programme in kinetic theory*, Inventiones, 2013
- Probabilistic approach
  - S. Mischler, C. Mouhot. *Kac's programme in kinetic theory*, Inventiones, 2013
  - D. Heydecker. *Convergence of the hard spheres Kac Process*. AOP, 2019
  - A. Durmus, A. Eberle
  - Series of papers by P. Dobson,J. Sanz-Serna, K. Zygalakis