Randomized lattice rules

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Lattice rules

Deterministic lattice rules

For $f \in \mathcal{H}_\alpha$ approximate the d-dimensional integral

$$I(f) := \int_{[0,1]^d} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

by an n-point lattice rule with generating vector $\boldsymbol{z} \in \mathbb{Z}_n^d$

$$Q_{n,\boldsymbol{z}}(f) := \frac{1}{n} \sum_{k \in \mathbb{Z}_n} f\left(\frac{\boldsymbol{z}k \bmod n}{n}\right).$$

Worst-case error for $f \in \mathcal{H}_\alpha$ for a given algorithm Q_n (e.g. $Q_{n,\boldsymbol{z}}$):

$$e^{\mathrm{det}}(Q_n, \mathcal{H}_\alpha) := \sup_{\substack{f \in \mathcal{H}_\alpha \\ \|f\|_\alpha \leq 1}} |I(f) - Q_n(f)|.$$

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$\rightsquigarrow$ For good lattice rule $Q_{n,\boldsymbol{z}}$ converges like $n^{-\alpha} \, \|f\|_\alpha$.
(Optimal. Bakhvalov.)

Randomized lattice rules

Consider a random family of deterministic rules $Q_n^* := \{Q_n^\omega\}_\omega$.

Randomized error or worst-case expected error for $f \in \mathcal{H}_\alpha$:

$$e^{\mathrm{ran}}(Q_n^*, \mathcal{H}_\alpha) := \sup_{\substack{f \in \mathcal{H}_\alpha \\ \|f\|_\alpha \leq 1}} \mathbb{E}_\omega\big[\,|I(f) - Q_n^\omega(f)|\,\big].$$

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How?
Random shifting? Random generating vector? Random n?
What is necessary?

Function space

Korobov space of dominating mixed smoothness $\alpha > 0$:

$$\mathcal{H}_\alpha := \left\{ f \in L_2([0,1]^d) : \|f\|_\alpha^2 := \sum_{\boldsymbol{h} \in \mathbb{Z}^d} r_\alpha^2(\boldsymbol{h}) \, |\hat{f}(\boldsymbol{h})|^2 < \infty \right\},$$

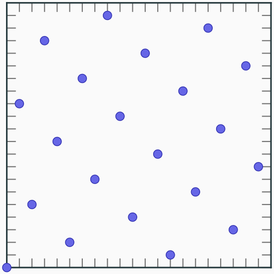
with

$$r_\alpha(\boldsymbol{h}) := \gamma_{\mathrm{supp}(\boldsymbol{h})}^{-1} \prod_{j \in \mathrm{supp}(\boldsymbol{h})} |h_j|^\alpha.$$

Weighted spaces... (Sloan, Woźniakowski...)

Example of a good lattice rule

Example: $n = 21$ and $\boldsymbol{z} = (1, 13)$ (Fibonacci lattice rule)



Constructive methods for deterministic error:

fast component-by-component (Nuyens & Cools 2006, ...)

$\rightarrow$ Fixed vector $\boldsymbol{z}$ for a given n.

(Or sequence of $n = p^m$, Cools, Kuo & Nuyens 2006).

Deterministic error

For $f \in \mathcal{H}_\alpha$, with $\alpha > 1/2$, and

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{h} \in \mathbb{Z}^d} \hat{f}(\boldsymbol{h})\, \mathrm{e}^{2\pi \mathrm{i}\, \boldsymbol{h} \cdot \boldsymbol{x}}, \qquad \hat{f}(\boldsymbol{h}) := \int_{[0,1]^d} f(\boldsymbol{x})\, \mathrm{e}^{-2\pi \mathrm{i}\, \boldsymbol{h} \cdot \boldsymbol{x}}\, \mathrm{d}\boldsymbol{x},$$

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we have

$$\frac{1}{n}\sum_{k\in\mathbb{Z}_n} f\left(\frac{\boldsymbol{z}k \bmod n}{n}\right) - \int_{[0,1]^d} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \sum_{\substack{0\neq\boldsymbol{h}\in\mathbb{Z}^d \\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0 \pmod{n}}} \hat{f}(\boldsymbol{h}),$$

by the character sum for $\mathbb{Z}_n$, we have for $a = \boldsymbol{z}\cdot\boldsymbol{h} \in \mathbb{Z}$,

$$\frac{1}{n}\sum_{k\in\mathbb{Z}_n} \exp(2\pi\mathrm{i}\, k\, a/n) = \mathbb{1}\{a \equiv 0 \pmod{n}\}.$$

The good set

Define the "good set" of generating vectors for a prime p as

$$G^{(p)} := \left\{ \boldsymbol{z} \in \mathbb{Z}_p^d : e^{\det}(Q_{p,\boldsymbol{z}}) \leq \inf_{\lambda \in [1/2, \alpha)} \left(\frac{4}{p} \sum_{0 \neq \boldsymbol{h} \in \mathbb{Z}^d} r_\alpha^{-1/\lambda}(\boldsymbol{h}) \right)^\lambda \right\}.$$

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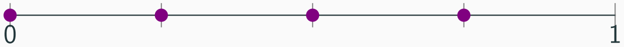
This set has more than $\left\lceil \frac{1}{2} p^d \right\rceil$ elements due to

$$\frac{1}{p^d} \sum_{\boldsymbol{z} \in \mathbb{Z}_p^d} \left[e^{\det}(Q_{p,\boldsymbol{z}}) \right]^{1/\lambda} \leq \frac{2}{p} \sum_{0 \neq \boldsymbol{h} \in \mathbb{Z}^d} r_\alpha^{-1/\lambda}(\boldsymbol{h}), \quad \forall \lambda \in [1/2, \alpha),$$

and Markov's inequality.

One-dimensional intuition

- "Lattice points": $x_k = k/n$, $k \in \mathbb{Z}_n$, for $n = 4, 5, 6$:



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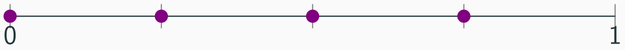

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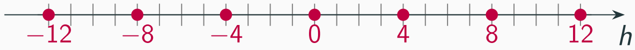

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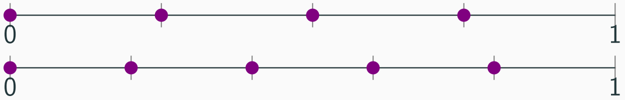

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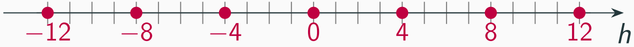

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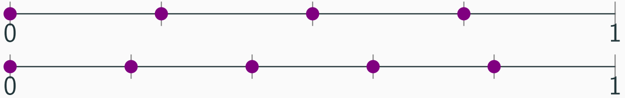

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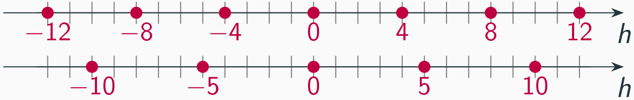

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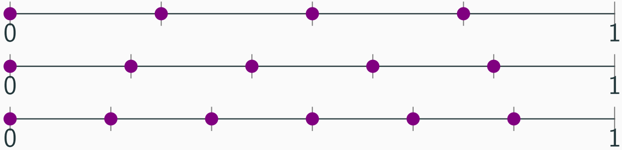

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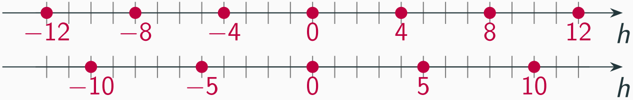

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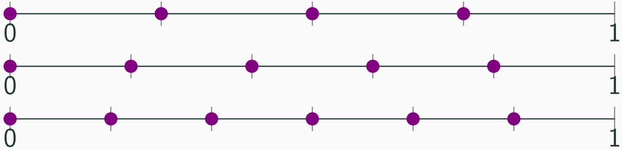

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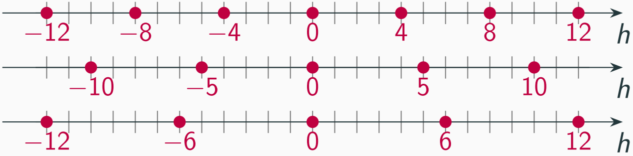

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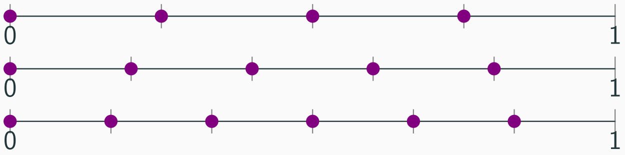

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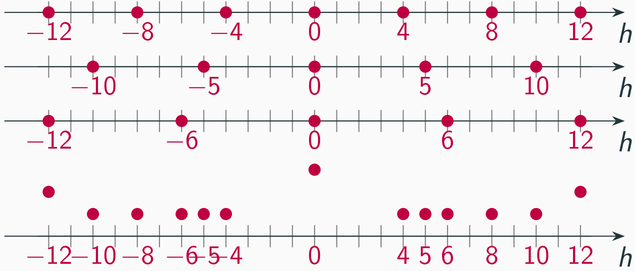
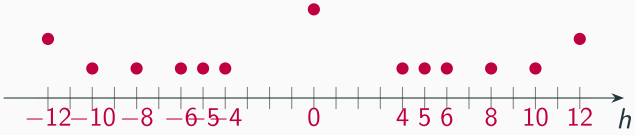


Zoom out a bit



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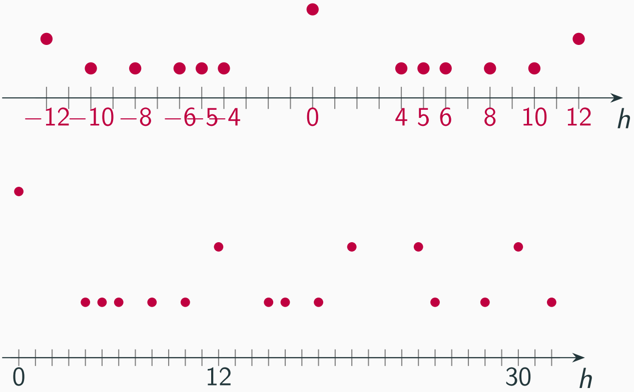

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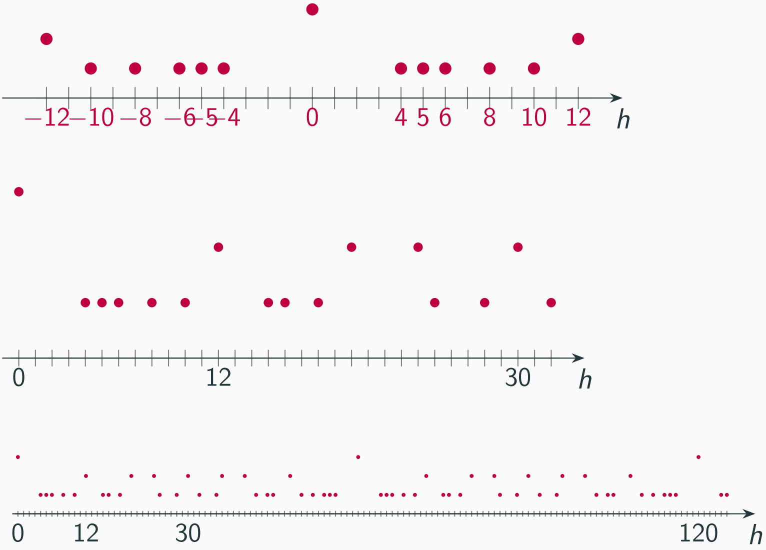

Prior art

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- Bakhvalov (1961): lower and upper bounds using lattice rules for randomized error.
- Kritzer, Kuo, Nuyens, M. Ullrich (2019): randomised algorithm using lattice rules to achieve the near optimal rate.

Algorithm 1 [KKNU19]

Uniformly sample a prime $p \in P_n$.
Uniformly sample a generating vector $\boldsymbol{z} \in G^{(p)}$.
Use the lattice rule with generating vector $\boldsymbol{z}$ and p sample points.

Modifying the good set to allow for CBC construction

Define

$$\widetilde{G}_{d,\boldsymbol{z}'}^{(p)} := \left\{ z_d \in \mathbb{Z}_p : \theta_{d,\boldsymbol{z}'}^{(p)}(z_d) \leq \inf_{\lambda \in [1/2, \alpha)} \left(\frac{4}{p} \sum_{\substack{0 \neq \boldsymbol{h} \in \mathbb{Z}^d \\ h_d \neq 0}} r_\alpha^{-1/\lambda}(\boldsymbol{h}) \right)^{2\lambda} \right\}.$$

This has more than $\left\lceil \frac{1}{2} p \right\rceil$ elements by a similar method to before.

Depends on the previously fixed values of $\boldsymbol{z}' = (z_1, \ldots, z_{d-1})$.

CBC randomised algorithm

Dick, Goda and Suzuki (2022): component-by-component method.

Algorithm 2 [DGS22 / ...]

Uniformly sample a prime $p \in P_n$.

for $j = 1$ **to** d **do**

  Uniformly sample $z_j \in \widetilde{G}^{(p)}_{d,\boldsymbol{z}'}$.

end for

Use the lattice rule with generating vector $\boldsymbol{z}$ and p sample points.

Existence of a fixed vector method

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Define the algorithm $K^*_{n,\boldsymbol{z}}$:

Algorithm 5 Fixed vector random algorithm (Kuo, Nuyens, Wilkes)

Uniformly sample $p \in P_n$.

Apply the lattice rule with the predefined $\boldsymbol{z}$ and p sample points.

Existence result

Theorem (Kuo, Nuyens, Wilkes)
There exists a vector $\boldsymbol{z} \in \mathbb{Z}^d$ which achieves the bound

$$e^{\mathrm{ran}}(K^*_{n,\boldsymbol{z}}) \leq \frac{C_\lambda \sqrt{\ln(n)}}{n^{\lambda+1/2}} \left(\sum_{0 \neq \boldsymbol{h} \in \mathbb{Z}^d} r_\alpha^{-1/\lambda}(\boldsymbol{h}) \right)^\lambda$$

for all $\frac{1}{2} \leq \lambda < \alpha$.

For the proof:
We take $\boldsymbol{z} \in \mathbb{Z}_N^d$ with $N := \prod_{p \in P_n} p$.
We average over all vectors which are good in the deterministic sense for all of the primes.

Caveats

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2. Existence, but what about construction?

CBC construction of the vector

What about the usual method?

We follow the standard CBC approach. If z_d is a component yet to be fixed, we can write

$$\left[e_d^{\text{ran}}(K_{n,\boldsymbol{z}}^*)\right]^2 = \left[e_{d-1}^{\text{ran}}(K_{n,\boldsymbol{z}'}^*)\right]^2 + \Theta(z_d).$$

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If we were to try to minimise $\Theta(z_d)$ at each dimension, we would have to search all possibilities that z_d could take.

This would be an $O(dn^{n+3})$ algorithm!

A detour...

Instead, we define a quantity $T^{(p)}(z_d^{(p)})$ which satisfies

$$\Theta(z_d) = \frac{1}{|P_n|^2} \sum_{p \in P_n} T^{(p)}(z_d^{(p)}).$$

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This allows us to fix the residues of the component z_d modulo each of the primes in P_n in increasing order. This uniquely sets the value of $z_d \in \mathbb{Z}_N$.

Constructing the vector

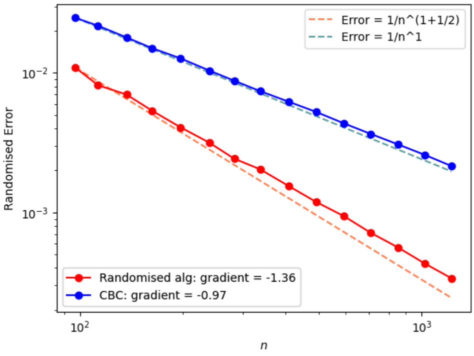
Algorithm 6 Optimal vector construction at n (Kuo, Nuyens, Wilkes)

for $j = 1$ to d **do**
  for $p \in P_n$ in increasing order **do**
    Compute $\theta_j^{(p)}(z_j^{(p)})$ for all $z_j^{(p)} \in \mathbb{Z}_p$.
    Compute $T_j^{(p)}(z_j^{(p)})$ for all $z_j^{(p)} \in \mathbb{Z}_p$.
    Choose from the $\lceil \tau p \rceil$ best choices for $\theta_j^{(p)}$ to minimize $T_j^{(p)}$.
  end for
end for

- Calculating the randomised error of an arbitrary vector takes $O(dn^4 \ln(n)^{-2})$.
- The complexity of this construction algorithm is only $O(dn^4)$ for product weights.

Randomised error vs deterministic error for $\alpha = 1$, $d = 30$

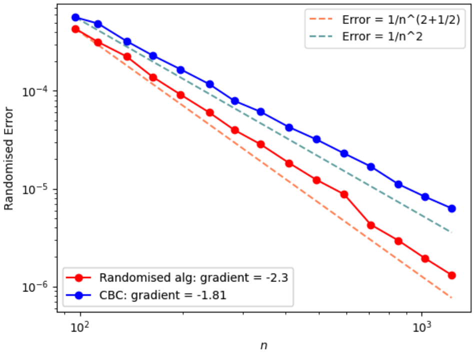
We use product weights $\gamma = \{j^{-3}\}_{j=1}^{d}$.



- The deterministic algorithm is $Q_{n,\boldsymbol{z}}$ for $\boldsymbol{z}$ chosen by CBC.
- The randomised algorithm is $K^*_{n,\boldsymbol{z}}$ with $\boldsymbol{z}$ chosen via the described method.

Randomised error vs deterministic error for $\alpha = 2$, $d = 30$

We use product weights $\gamma = \{j^{-3}\}_{j=1}^{d}$.



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Conclusions

- Fixed vector algorithm:

Theorem
For $\alpha > 1/2$ *and all* $\lambda \in [1/2, \alpha)$*:*

$$e^{\mathrm{ran}}(K^*_{n,\boldsymbol{z}}) \leq \frac{(C_{\tau,\lambda} \ln n)^{1/2}}{n^{\lambda+1/2}} (\mu_{d,\alpha,\boldsymbol{\gamma}}(\lambda))^{\lambda}.$$

- For $\alpha \in (0, 1/2]$: the usual trick does not work since we want a fixed vector $\boldsymbol{z}$.
- Solved by relaxing the sup in the error bound:

Theorem
For $\alpha > 0$ *and all* $\lambda \in (0, \alpha)$*,* $r \in \mathbb{N}$ *and* $r \geq 1/(2\lambda)$*:*

$$e^{\mathrm{rms}}(K^{**}_{n,\boldsymbol{z}}) \leq \frac{1}{n^{\lambda+(r-1)/(2r)}} \left(\frac{C_3 \, r \ln(n)}{C_1 \, \ln(r+1)} \right)^{1/2} (\mu_{d,\alpha,\boldsymbol{\gamma}}(\lambda))^{\lambda}.$$

Thanks for listening!