Multilevel methods for kernel interpolation in uncertainty quantification

Abi Srikumar

Joint work with Alec Gilbert, Mike Giles, Frances Kuo and Ian Sloan

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Introduction

What is uncertainty quantification?
We try to better understand models with intrinsic randomness to them.

We will look at multilevel kernel-based lattice-point approximation for PDE solutions with random coefficients where the diffusion coefficient is periodic in its stochastic variables.

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We will look at multilevel kernel-based lattice-point approximation for PDE solutions with random coefficients where the diffusion coefficient is periodic in its stochastic variables.

Note this is work in progress!!

Problem setting

Consider the stochastic PDE over bounded, convex domain $D \subset \mathbb{R}^d$,

$$
\begin{aligned}
-\nabla \cdot (a(\boldsymbol{x}, \boldsymbol{y}) \nabla u(\boldsymbol{x}, \boldsymbol{y})) &= f(\boldsymbol{x}) & \boldsymbol{x} &\in D, \\
u(\boldsymbol{x}, \boldsymbol{y}) &= 0, & \boldsymbol{x} &\in \partial D,
\end{aligned}
$$

where $f \in L^2(D)$ is deterministic. Our random coefficient is now periodic in our stochastic variables,

$$
a(\boldsymbol{x}, \boldsymbol{y}) = a_0(\boldsymbol{x}) + \sum_{j \geq 1} \sin(2\pi y_j)\, \psi_j(\boldsymbol{x})
$$

where $y_j \sim \mathrm{Unif}[-\frac{1}{2}, \frac{1}{2}]$ and $a_0, \psi_j \in L^\infty(D)$ for $j \geq 1$ such that $\sum_{j \geq 1} |\psi_j(\boldsymbol{x})| < \infty$ for any $\boldsymbol{x} \in D$.

Kaarnioja, Kuo and Sloan (2020)

Problem setting

Our assumptions

1. $a_0 \in L^\infty(D), \psi_j \in L^\infty(D)$ for all $j \geq 1$ and $\sum_{j\geq 1} \|\psi_j\|_{L^\infty} < \infty$
2. $0 < a_{\min} \leq a(\boldsymbol{x}, \boldsymbol{y}) \leq a_{\max}$ for all $\boldsymbol{x} \in D$ and $\boldsymbol{y} \in \Omega$
3. $\sum_{j\geq 1} \|\psi_j\|_{L^\infty}^p < \infty$ for some $p \in (0, 1)$
4. $a_0 \in W^{1,\infty}(D)$ and $\sum_{j\geq 1} \|\psi_j\|_{W^{1,\infty}(D)} < \infty$ where $\|v\|_{W^{1,\infty}(D)} := \max\{\|v\|_{L^\infty}, \|\nabla v\|_{L^\infty}\}$
5. $\|\psi_1\|_{L^\infty} \geq \|\psi_2\|_{L^\infty(D)} \geq \cdots$
6. D is a convex bounded polyhedron with plane faces

Kaarnioja, Kuo and Sloan (2020)

The plan

Considerations:

- Our field is infinite dimensional
- For approximation, we require the target function to be pointwise well-defined in both $\boldsymbol{x}$ and $\boldsymbol{y}$
- Approximate the solution

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The plan

Considerations:

- Our field is infinite dimensional $\rightarrow$ Truncate the dimension
- For approximation, we require the target function to be pointwise well-defined in both $\boldsymbol{x}$ and $\boldsymbol{y}$ $\rightarrow$ Finite element method
- Approximate the solution $\rightarrow$ Multilevel kernel interpolation

The weighted Korobov space

We are interested in the Hilbert space $H_{\alpha,\boldsymbol{\gamma}}$ of one-periodic L^2 functions with defined on $[0,1]^s$ with absolutely convergent Fourier series.

The norm and inner product of $H_{\alpha,\boldsymbol{\gamma}}$ are

$$\|f\|_{s,\alpha,\boldsymbol{\gamma}}^2 := \sum_{\boldsymbol{h}\in\mathbb{Z}^s} \left|\widehat{f}(\boldsymbol{h})\right|^2 r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h}) \quad \text{and} \quad \langle f, g\rangle_{\alpha,\boldsymbol{\gamma}} := \sum_{\boldsymbol{h}\in\mathbb{Z}^d} \widehat{f}(\boldsymbol{h})\,\overline{\widehat{g}(\boldsymbol{h})}\, r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h}),$$

with

$$r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h}) := \frac{1}{\gamma_{\mathrm{supp}(\boldsymbol{h})}} \prod_{j\in\mathrm{supp}(\boldsymbol{h})} |h_j|^{2\alpha}.$$

For integer α, the norm can be written as

$$\|f\|_{H_{\alpha,\boldsymbol{\gamma}}} = \sqrt{\sum_{\mathfrak{u}\subseteq\{1:s\}} \frac{1}{(2\pi)^{2\alpha|\mathfrak{u}|}\gamma_{\mathfrak{u}}} \int_{[0,1]^{|\mathfrak{u}|}} \left| \int_{[0,1]^{s-|\mathfrak{u}|}} \left(\prod_{j\in\mathfrak{u}} \frac{\partial^\alpha}{\partial y_j^\alpha}\right) f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}_{-\mathfrak{u}} \right|^2 \mathrm{d}\boldsymbol{y}_{\mathfrak{u}}}.$$

The weighted Korobov space

$H_{\alpha,\gamma}$ is a reproducing kernel Hilbert space with kernel

$$K(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{h}\in\mathbb{Z}^s} \frac{e^{2\pi \boldsymbol{h}\cdot(\boldsymbol{x}-\boldsymbol{y})}}{r_{\alpha,\gamma}(\boldsymbol{h})},$$

which satisfies

1. $K(\boldsymbol{x},\boldsymbol{y}) = K(\boldsymbol{y},\boldsymbol{x})$ for all $\boldsymbol{x},\boldsymbol{y} \in [0,1]^s$,
2. $K(\cdot,\boldsymbol{y}) \in H_{\alpha,\gamma}$ for all $\boldsymbol{y} \in [0,1]^s$,
3. $\left\langle f, K(\cdot,\boldsymbol{y})\right\rangle_{\alpha,\gamma} = f(\boldsymbol{y})$ for all $f \in H_{\alpha,\gamma}$ and all $\boldsymbol{y} \in [0,1]^s$.

Rank-1 lattices

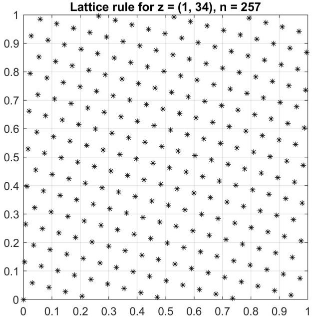
Lattice rules belong to *Quasi-Monte Carlo (QMC) methods*, cleverly designed integration techniques that are less susceptible to the curse of dimensionality.

An n-point rank-1 lattice is given by

$$\boldsymbol{t}_k = \frac{k\boldsymbol{z} \bmod n}{n} \qquad \text{for } k = 0, \ldots, n-1$$

where $\boldsymbol{z} \in \mathbb{U}_n^s$ is the generating vector.

The generating vector is constructed using a *component-by-component* algorithm.



Sloan and Joe (1994)

Lattice-based kernel approximation

We approximate $f \in H_{\alpha,\boldsymbol{\gamma}}$ by the kernel interpolant of the form

$$I_n(f)(\boldsymbol{y}) := \sum_{k=0}^{n-1} a_k\, K(\boldsymbol{t}_k, \boldsymbol{y}) \qquad \text{for} \quad \boldsymbol{y} \in [0,1]^s.$$

which interpolates f at n rank-1 lattice points.

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That is,

$$I_n(f)(\boldsymbol{t}_\ell) = f(\boldsymbol{t}_\ell) \quad \text{for all } \ell = 0, \ldots, n-1.$$

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That is,

$$I_n(f)(\boldsymbol{t}_\ell) = f(\boldsymbol{t}_\ell) \quad \text{for all } \ell = 0, \ldots, n-1.$$

The coefficients a_k are obtained by solving the resulting linear system,

$$f(\boldsymbol{t}_\ell) = \sum_{k=0}^{n-1} a_k \, K(\boldsymbol{t}_k, \boldsymbol{t}_\ell) \quad \text{for all } \ell = 0, \ldots, n-1.$$

Lattice-based kernel approximation

Note that:

1. Kernel interpolant is optimal for given function values.

Lattice-based kernel approximation

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2. Following from properties of the kernel and lattice structure, we have a circulant matrix $\rightarrow$ fast Fourier Transform with cost $\mathcal{O}(n \log n)$

Lattice-based kernel approximation

Note that:

1. Kernel interpolant is optimal for given function values.
2. Following from properties of the kernel and lattice structure, we have a circulant matrix $\rightarrow$ fast Fourier Transform with cost $\mathcal{O}(n \log n)$
3. Aim is to construct the interpolant at a cheaper cost by leveraging multilevel methods

Multilevel methods

We want to estimate $\mathbb{E}[P]$, the easiest way is to take a direct average over a number of samples

$$\mathbb{E}[P] \approx \frac{1}{N} \sum_{n=1}^{N} P^{(n)}.$$

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Given a sequence $P_0, \ldots, P_{L-1}$ which approximates P_L with increasing accuracy, we have that

$$\mathbb{E}[P_L] = \mathbb{E}[P_0] + \sum_{\ell=1}^{L} \mathbb{E}[P_\ell - P_{\ell-1}].$$

This can be estimated by

$$\frac{1}{N_0} \sum_{n=1}^{N_0} P_0^{(0,n)} + \sum_{\ell=0}^{L} \frac{1}{N_\ell} \sum_{n=1}^{N_\ell} \left(P_\ell^{(\ell,n)} - P_{\ell-1}^{(\ell,n)} \right).$$

Related work

- Multilevel methods
  Giles (2008), Giles and Waterhouse (2009), Heinrich (1998), Heinrich (2001)

- Uncertainty quantification
  Cliffe, Giles, Scheichl and Teckentrup (2011), Gilbert and Scheichl (2020), Hakula et al. (2023), Kaarnioja, Kuo and Sloan (2020), Kuo, Schwab and Sloan (2015)

- Kernel-based approximation
  Belhadji, Bardenet and Chainais (2020), Kaarnioja et al. (2022), Schaback (1995), Schaback and Wendland (2006), Zeng, Kritzer and Hickernell (2009), Zeng, Leung and Hickernell (2006)

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Today: Theory for multilevel kernel interpolation !

Formulation of multilevel kernel interpolant

We begin by defining

$$u_\ell := u_{h_\ell}^{s_\ell}$$

as the finite element solution at level ℓ with mesh width h_ℓ and dimension truncated to s_ℓ.

We also use the notation $I_\ell := I_{n_\ell}$ to indicate the kernel interpolant constructed using n_ℓ lattice points.

Formulation of multilevel kernel interpolant

For $\ell = 0, 1, \ldots$, consider a sequence of kernel interpolants I_ℓ using a decreasing number of points $n_0 > n_1 > \cdots$ and a sequence of finite element approximations u_ℓ using an increasing number of finite element nodes, i.e., $h_0 > h_1 > \cdots$.

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For $\ell = 0, 1, \ldots$, consider a sequence of kernel interpolants I_ℓ using a decreasing number of points $n_0 > n_1 > \cdots$ and a sequence of finite element approximations u_ℓ using an increasing number of finite element nodes, i.e., $h_0 > h_1 > \cdots$.

Now, the approximation at the maximum level $L \in \mathbb{N}$ is given by

$$I_L u_L := I_0 u_0 + \sum_{\ell=1}^{L} I_\ell(u_\ell - u_{\ell-1}).$$

Error breakdown

The total approximation error can be broken down as follows:

$$
\begin{aligned}
u - I_L u_L &= u - \sum_{\ell=0}^{L} I_\ell (u_\ell - u_{\ell-1}) \\
&= u - u_{h_L}^{s_L} + \sum_{\ell=0}^{L} (I - I_\ell)(u_\ell - u_{\ell-1}) \\
&= \underbrace{u - u^{s_L}}_{\text{DT error}} + \underbrace{u^{s_L} - u_{h_L}^{s_L}}_{\text{FE error}} + \underbrace{\sum_{\ell=0}^{L} (I - I_\ell)(u_\ell - u_{\ell-1})}_{\text{ML KI error}}
\end{aligned}
$$

Error breakdown

The total error can be expressed as

$$
\begin{aligned}
&\sqrt{\int_\Omega \int_D (u(\boldsymbol{x},\boldsymbol{y}) - I_L u(\boldsymbol{x},\boldsymbol{y}))^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}} \\
&\leq \sqrt{\int_\Omega \int_D (u(\boldsymbol{x},\boldsymbol{y}) - u^{s_L}(\boldsymbol{x},\boldsymbol{y}))^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}} \\
&\qquad + \sqrt{\int_\Omega \int_D (u^{s_L}(\boldsymbol{x},\boldsymbol{y}) - u^{s_L}_{h_L}(\boldsymbol{x},\boldsymbol{y}))^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}} \\
&\qquad\qquad + \sum_{\ell=0}^{L} \sqrt{\int_\Omega \int_D ((I - I_\ell)(u_\ell - u_{\ell-1})(\boldsymbol{x},\boldsymbol{y}))^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}}
\end{aligned}
$$

Dimension truncation error

Theorem (Kaarnioja et al. (2022))

Suppose the PDE problem satisfies the required conditions. Then for any $s \in \mathbb{N}$*, there exists a constant* $C > 0$ *such that*

$$\sqrt{\int_\Omega \int_D (u(\boldsymbol{x}, \boldsymbol{y}) - u^s(\boldsymbol{x}, \boldsymbol{y}))^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}} \leq C\, s^{-\frac{1}{p}+\frac{1}{2}}\, \|f\|_{H^{-1}(D)},$$

where the constant $C > 0$ *is independent of* s *and* f*.*

Finite element error

Theorem (Kaarnioja et al. (2022))

Under the required assumptions, for every $\boldsymbol{y} \in \Omega$ and $f \in L^2(D)$, the following asymptotic convergence estimate holds

$$\|u(\cdot, \boldsymbol{y}) - u_h(\cdot, \boldsymbol{y})\|_{L^2(D)} \leq C\, h^2 \, \|f\|_{L^2(D)} \quad \text{as } h \to 0,$$

where the constant $C > 0$ is independent of h and $\boldsymbol{y}$.

Finite element error

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where the constant $C > 0$ is independent of h and $\boldsymbol{y}$.

Applying to dimension truncated problem, we have

$$\sqrt{\int_\Omega \int_D (u^s(\boldsymbol{x}, \boldsymbol{y}) - u_h^s(\boldsymbol{x}, \boldsymbol{y}))^2 \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{y}} \leq C\, h^2\, \|f\|_{L^2(D)}.$$

Multilevel error

We now bound the Multilevel kernel interpolant component of error. Recall $u_\ell = u_{h_\ell}^{s_\ell}$.

$$
\begin{aligned}
\sum_{\ell=0}^{L} & \sqrt{\int_{\Omega_\ell} \int_D ((I - I_\ell)(u_\ell - u_{\ell-1})(\boldsymbol{x}, \boldsymbol{y}))^2 \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}\boldsymbol{y}} \\
&= \sqrt{\int_{\Omega_0} \int_D ((I - I_0)(u_0)(\boldsymbol{x}, \boldsymbol{y}))^2 \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}\boldsymbol{y}} \\
&\qquad + \sum_{\ell=1}^{L} \sqrt{\int_{\Omega_\ell} \int_D ((I - I_\ell)(u_\ell - u_{\ell-1})(\boldsymbol{x}, \boldsymbol{y}))^2 \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}\boldsymbol{y}},
\end{aligned}
$$

Kernel interpolation error

We define the worst-case error of approximation wrt the L^2-norm as

$$e^{\text{wor}}(I_n, L_2) := \sup_{\|f\|_{H_{\alpha,\gamma}} < 1} \|f - I_n(f)\|_{L^2}.$$

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We define the worst-case error of approximation wrt the L^2-norm as

$$e^{\mathrm{wor}}(I_n, L_2) := \sup_{\|f\|_{H_{\alpha,\gamma}} < 1} \|f - I_n(f)\|_{L^2}.$$

Following from the optimality of the kernel interpolant, we have

$$e^{\mathrm{wor}}(I_n, L_2) \leq \mathcal{S}_n(\boldsymbol{z}).$$

$\mathcal{S}_n(\boldsymbol{z})$ (which bounds the trig polynomial approximation) is shown in Cools, Kuo, Nuyens and Sloan (2021) and Kuo, Mo and Nuyens (2023+) to have convergence $\mathcal{O}(n^{-\alpha/2+\delta})$ for $\delta > 0$.

Kernel interpolation error

Theorem (Kaarnioja et al. (2022))

Given $s \geq 1$, $\alpha > 1/2$ and weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subset \mathbb{N}}$, a lattice-based kernel interpolant I_n can be constructed such that

$$e^{\mathrm{wor}}(I_n, L_2) \leq C_{\boldsymbol{\gamma},\delta}\, n^{-\frac{\alpha}{2}+\delta} \qquad \text{for all } \delta \in (0, \alpha/2)$$

where the implied constant depends on α but is independent of s.

Multilevel error

Now, considering a term in the multilevel error component,

$$
\begin{aligned}
&\sqrt{\int_{\Omega_\ell}\int_D ((I - I_\ell)(u_\ell - u_{\ell-1})(\boldsymbol{x},\boldsymbol{y}))^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}} \\
\leq &\sqrt{\int_D\int_{\Omega_\ell} ((I - I_\ell)(u_\ell - u_{\ell-1})(\boldsymbol{x},\boldsymbol{y}))^2 \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}} \\
\leq &\sqrt{\int_D \|(I - I_\ell)(u_\ell - u_{\ell-1})(\boldsymbol{x},\cdot)\|_{L^2(\Omega_\ell)}^2 \,\mathrm{d}\boldsymbol{x}} \\
\leq &e^{\mathrm{wor}}(I_\ell, L^2(\Omega_\ell))\sqrt{\int_D \|(u_\ell - u_{\ell-1})(\boldsymbol{x},\cdot)\|_{H_{\alpha,\gamma}(\Omega_\ell)}^2 \,\mathrm{d}\boldsymbol{x}}
\end{aligned}
$$

Multilevel error

$$
\begin{aligned}
&e^{\mathrm{wor}}(I_\ell, L^2(\Omega_\ell))\sqrt{\int_D \|(u_{h_\ell}^{s_\ell} - u_{h_{\ell-1}}^{s_{\ell-1}})(\boldsymbol{x},\cdot)\|^2_{H_{\alpha,\boldsymbol{\gamma}}(\Omega_\ell)}\,\mathrm{d}\boldsymbol{x}} \\
&\leq e^{\mathrm{wor}}(I_\ell, L^2(\Omega_\ell))\Bigg[\sqrt{\int_D \|(u_{h_{\ell-1}}^{s_\ell} - u_{h_{\ell-1}}^{s_{\ell-1}})(\boldsymbol{x},\cdot)\|^2_{H_{\alpha,\boldsymbol{\gamma}}(\Omega_\ell)}\,\mathrm{d}\boldsymbol{x}} \\
&+ \sqrt{\int_D \|(u^{s_\ell} - u_{h_\ell}^{s_\ell})(\boldsymbol{x},\cdot)\|^2_{H_{\alpha,\boldsymbol{\gamma}}(\Omega_\ell)}\,\mathrm{d}\boldsymbol{x}} + \sqrt{\int_D \|(u^{s_\ell} - u_{h_{\ell-1}}^{s_\ell})(\boldsymbol{x},\cdot)\|^2_{H_{\alpha,\boldsymbol{\gamma}}(\Omega_\ell)}\,\mathrm{d}\boldsymbol{x}}\ \Bigg]
\end{aligned}
$$

Theorem (Gilbert, Giles, Kuo, Sloan and S. (In prep.))

Suppose the PDE problem satisfies the required assumptions. For $\alpha \geq 1$, $f \in L^2(D)$ and $b_i := \frac{\|\psi_j\|_{L^\infty}}{a_{\min}}$, the weight parameters $(\gamma_{\mathfrak{u}})_{\mathfrak{u}\subset\mathbb{N}}$ can be chosen such that

$$\sqrt{\int_D \|(u_h^{s_\ell} - u_h^{s_{\ell-1}})(\boldsymbol{x}, \cdot)\|^2_{H_{\alpha,\gamma}(\Omega_\ell)} \,\mathrm{d}\boldsymbol{x}}$$

$$\leq \frac{C\,\|f\|_{H^{-1}(D)}}{s^{\min(\frac{1}{p}-1,\eta)}} \sqrt{\sum_{\mathfrak{u}\subseteq\{1:s_\ell\}} \frac{1}{\gamma_{\mathfrak{u}}} \Bigg(\sum_{\boldsymbol{m}\in\{1:\alpha\}^{|\mathfrak{u}|}} (|\boldsymbol{m}|+k)! \prod_{i\in\mathfrak{u}} b_i^{m_i} S(\alpha, m_i) \Bigg)^2},$$

where $\eta > 0$, $k \geq \alpha$ and $C > 0$ is independent of s_ℓ and $s_{\ell-1}$.

Theorem (Gilbert, Giles, Kuo, Sloan and S. (In prep.))

Suppose the PDE problem satisfies the required assumptions. For $\alpha \geq 1$, weight parameters $(\gamma_{\mathfrak{u}})_{\mathfrak{u} \subset \mathbb{N}}$, $f \in L^2(D)$ and defining $\overline{b}_i := \frac{\|\nabla \psi_j\|_{L^\infty}}{a_{\min}}$, the following estimate holds

$$\sqrt{\int_D \|(u^s - u_h^s)(\boldsymbol{x}, \cdot)\|^2_{H_{\alpha,\boldsymbol{\gamma}}(\Omega)} \, \mathrm{d}\boldsymbol{x}}$$

$$\leq C\, h^2 \, \|f\|_{L^2(D)} \sqrt{\sum_{\mathfrak{u} \subseteq \{1:s\}} \frac{1}{\gamma_{\mathfrak{u}}} \Bigg(\sum_{\boldsymbol{m} \in \{1:\alpha\}^{|\mathfrak{u}|}} (|\boldsymbol{m}| + 5)! \prod_{i \in \mathfrak{u}} \overline{b}_i^{m_i} S(\alpha, m_i) \Bigg)^2},$$

where $C > 0$ is independent of h.

Sketch of the proof:

1. Cauchy-Schwarz and Fubini's theorem
2. We need bounds for $\|\partial^{\boldsymbol{\nu}}(u - u_h)(\cdot, \boldsymbol{y})\|_{L^2(D)}$
3. **Theorem.**

$$\|\partial^{\boldsymbol{\nu}}(u - u_h)\|_{H_0^1(D)} \leq C\, h\, \|f\|_{L^2}\, (2\pi)^{|\boldsymbol{\nu}|} \sum_{\boldsymbol{m} \leq \boldsymbol{\nu}} (|\boldsymbol{m}| + 2)!\, \overline{\boldsymbol{b}}^{\boldsymbol{m}} \prod_{i \geq 1} S(\nu_i, m_i)$$

4. Aubin-Nitsche duality argument
5. **Theorem.**

$$\|\partial^{\boldsymbol{\nu}}(u - u_h)\|_{L^2(D)} \leq C\, h^2 \|f\|_{L^2}\, (2\pi)^{|\boldsymbol{\nu}|} \sum_{\boldsymbol{m} \leq \boldsymbol{\nu}} (|\boldsymbol{m}| + 5)!\, \overline{\boldsymbol{b}}^{\boldsymbol{m}} \prod_{i \geq 1} S(\nu_i, m_i)$$

Putting it together

$$\text{Error} \approx \text{DT error} + \text{FE error} + \text{ML KI error}$$

Theorem (Gilbert, Giles, Kuo, Sloan and S. (In prep.))

Suppose the PDE problem satisfies the required assumptions. For $\alpha \geq 1$, and $f \in L^2(D)$, the weight parameters $(\gamma_{\mathfrak{u}})_{\mathfrak{u} \subset \mathbb{N}}$ can be chosen such that

$$\sqrt{\int_\Omega \int_D (u(\boldsymbol{x}, \boldsymbol{y}) - I_L u(\boldsymbol{x}, \boldsymbol{y}))^2 \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}\boldsymbol{y}}$$

$$\leq C_{params} \left(s_L^{-\frac{1}{p}+\frac{1}{2}} + h_L^2 + \sum_{\ell=0}^{L} \frac{1}{n_\ell^{\frac{\alpha}{2}-\delta}} \left(h_{\ell-1}^2 + s_{\ell-1}^{-\min(\frac{1}{p}-1,\eta)} \right) \right)$$

Numerical results

TBA.

Conclusion

We construct a multilevel kernel interpolant in the hope of reducing the current cost.

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To do

- Full implementation and numerics of multilevel methodology
- Possibly improve multilevel dimension truncation error
- Multilevel approximation for other applications

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Thank you for your attention :)

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