Domain decomposition methods for SPDEs

Monika Eisenmann

joint work with Evelyn Buckwar, Ana Djurdjevac

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Lund University

Outline

Underlying problem

Operator splitting & Domain decomposition

Convergence result

Numerical example

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Underlying problem

Goal: Efficient numerical approximation of an SPDE

Abstract problem

$$\begin{cases} \mathrm{d}u(t) + A(t, u(t))\,\mathrm{d}t = F(t, u(t))\,\mathrm{d}t \\ \qquad\qquad\qquad\qquad\qquad + B(t, u(t))\,\mathrm{d}W(t), \quad t \in (0, T], \\ u(0) = u_0. \end{cases}$$

Underlying problem

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Abstract problem

$$\begin{cases} \mathrm{d}u(t) + A(t, u(t))\,\mathrm{d}t = F(t, u(t))\,\mathrm{d}t \\ \qquad\qquad\qquad\qquad\qquad\qquad + B(t, u(t))\,\mathrm{d}W(t), \quad t \in (0, T], \\ u(0) = u_0. \end{cases}$$

More concrete example

$$\begin{cases} \mathrm{d}u(t,x) - 0.1(1 + \exp(-t))\Delta u(t,x)\,\mathrm{d}t \\ \qquad\qquad = \sin(u(t,x))\,\mathrm{d}t + u(t,x)\,\mathrm{d}W(t,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in (0, T], x \in (0,1)^2, \\ u(t,x) = 0, \qquad\qquad\qquad\qquad\qquad\qquad t \in (0, T], x \in \partial(0,1)^2, \\ u(0,x) = u_0(x), \qquad\qquad\qquad\qquad\qquad x \in (0,1)^2. \end{cases}$$

Notation

- H is separable Hilbert space, e.g. $H = L^2(\mathcal{D})$,
- V is a separable, reflexive Banach space, e.g. $V = H_0^1(\mathcal{D})$,
- the spaces built a Gelfand triple $V \hookrightarrow H \cong H^* \hookrightarrow V^*$,
- initial data $u_0 \colon \Omega \to H$ belongs to $L^2(\Omega, H)$,
- Q-Wiener process W such that
  - Q being positive definite and self-adjoint,
  - $Q^{\frac{1}{2}} \in \mathcal{HS}(H)$.

Assumptions on the data, A

Assume that the family of operators $A(t,\cdot)\colon V \to V^*$ fulfills:

- monotone,
- coercive,
- ν-Hölder continuous in the t-component for $\nu \in (0, \frac{1}{2}]$,
- Lipschitz continuous in the second component.

Example: $A(t,\cdot) = -\Delta\colon H_0^1 \to H^{-1}$.

Assumptions on the data, F and B

Assume that the family of operators $F(t, \cdot)\colon H \to H$, $B(t, \cdot)\colon H \to \mathcal{HS}(Q^{\frac{1}{2}}(H), H)$ fulfill:

- $\|F(t,0)\|_H$ and $\|B(t,0)\|_{L_0^2}$ are bounded,
- ν-Hölder continuous in the first component for $\nu \in (0, \frac{1}{2}]$,
- Lipschitz continuous in the second component.

Example: $F(t, \cdot)\colon L^2(\mathcal{D}) \to L^2(\mathcal{D})$, $F(t, u) = \sqrt{t}\sin(u)$

First scheme

For step size h, a standard scheme can look as follows:

$$\begin{cases} U^n - U^{n-1} + hA(t_n, U^n) = hF(t_{n-1}, U^{n-1}) \\ \qquad\qquad\qquad\qquad\qquad + B(t_{n-1}, U^{n-1})\Delta W^{n,K} & \text{in } V^* \\ U^0 = u_0 & \text{in } H. \end{cases}$$

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Or equivalently

$$V^n = U^{n-1} + hF(t_{n-1}, U^{n-1}) + B(t_{n-1}, U^{n-1})\Delta W^{n,K}$$
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Note

- Use an implicit step for possible stiff operator.
- Use an explicit scheme for nonlinear part if possible.

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Literature

Operator splittings:

- Reaction-Diffusion problems: [Arrarás, in 't Hout, Hundsdorfer, Portero, 2017], [Hansen, Stillfjord, 2012], [Koch, Neuhauser, Thalhammer 2013], ...
- S(P)DEs: [Ableidinger, Buckwar, 2016], [Buckwar, et. al. 2022], [Bréhier, Goudenège, 2019], ...
- ...

Domain decomposition:

- Deterministic PDEs: [Arrarás, Portero, 2015], [E., Hansen, 2018, 2022], [Mathew, Polyakov, Russo, Wang, 1998], [Vabishechevich, 2008], ...
- Random PDEs: [Mu Wang, 2019], [Sarkar, Benabbou, Ghanem, 2008].

Advantages domain decomposition

Advantages

- Cheaper sub-problems
- Problems can be implemented in parallel

Plan

- So far: Domain decomposition for
  - deterministic time dependent PDEs,
  - random stationary PDEs
- Extend to time dependent SPDEs

Operator splitting

Consider the stochastic evolution equation

$$\mathrm{d}u(t) + A(t, u(t))\,\mathrm{d}t = B(t, u(t))\,\mathrm{d}W(t).$$

Operator splitting

Consider the stochastic evolution equation

$$\begin{aligned}\mathrm{d}u(t) + \big(A_1(t, u(t)) + A_2(t, u(t))\big)\,\mathrm{d}t \\ = \big(B_1(t, u(t)) + B_2(t, u(t))\big)\,\mathrm{d}W(t).\end{aligned}$$

Operator splitting

Consider the stochastic evolution equation

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Then we solve

$$\mathrm{d}u(t) + A_1(t, u(t))\,\mathrm{d}t = B_1(t, u(t))\,\mathrm{d}W(t)$$

and

$$\mathrm{d}u(t) + A_2(t, u(t))\,\mathrm{d}t = B_2(t, u(t))\,\mathrm{d}W(t)$$

individually and “glue” them together to a suitable solution.

Different splittings

For

$$\begin{cases} \mathrm{d}u(t) + A(t, u(t))\,\mathrm{d}t = B(t, u(t))\,\mathrm{d}W(t), \quad t \in (0, T], \\ u(0) = u_0, \end{cases}$$

we consider the solution operator $\mathcal{L}_{A,B}$ such that

$$\mathcal{L}_{A,B} u_0 = \mathcal{L}_{A_1+A_2,B_1+B_2} u_0 = u$$

- Lie splitting

$$\mathcal{L}_{A,B} \approx \mathcal{L}_{A_2,B_2} \circ \mathcal{L}_{A_1,B_1}$$

- Sum splitting

$$\mathcal{L}_{A,B} \approx \frac{1}{2}\big(\mathcal{L}_{2A_2,2B_2} + \mathcal{L}_{2A_1,2B_1}\big)$$

- Strang splitting

$$\mathcal{L}_{A,B} \approx \mathcal{L}_{\frac{1}{2}A_1,\frac{1}{2}B_1} \circ \mathcal{L}_{A_2,B_2} \circ \mathcal{L}_{\frac{1}{2}A_1,\frac{1}{2}B_1}$$

Domain decomposition integrators

Example: Stochastic heat equation

For

- $T > 0$,
- open, connected and bounded set $\mathcal{D} \subseteq \mathbf{R}^d$,

consider the problem

$$\begin{cases} \mathrm{d}u(t,x) - \Delta u(t,x)\,\mathrm{d}t = B(u)\,\mathrm{d}W(t,x), & (t,x) \in (0,T) \times \mathcal{D}, \\ u(t,x) = 0, & (t,x) \in (0,T) \times \partial\mathcal{D}, \\ u(0,x) = u_0(x), & x \in \mathcal{D}. \end{cases}$$

Suitable operators: Original problem

We find the weak formulation through

$$\int_{\mathcal{D}} \mathrm{d}uv \,\mathrm{d}x - \int_{\mathcal{D}} \big(\nabla \cdot \nabla u\big) v \,\mathrm{d}x \,\mathrm{d}t = \int_{\mathcal{D}} B(u) v \,\mathrm{d}x \,\mathrm{d}W$$

for $v \in C_0^\infty(\mathcal{D})$.

Suitable operators: Original problem

We find the weak formulation through

$$\int_{\mathcal{D}} \mathrm{d}uv \,\mathrm{d}x + \int_{\mathcal{D}} \nabla u \cdot \nabla v \,\mathrm{d}x \,\mathrm{d}t = \int_{\mathcal{D}} B(u)v \,\mathrm{d}x \,\mathrm{d}W$$

for $v \in C_0^\infty(\mathcal{D})$.

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We find the weak formulation through

$$\int_{\mathcal{D}} \mathrm{d}uv \, \mathrm{d}x + \int_{\mathcal{D}} \nabla u \cdot \nabla v \, \mathrm{d}x \, \mathrm{d}t = \int_{\mathcal{D}} B(u)v \, \mathrm{d}x \, \mathrm{d}W$$

for $v \in C_0^\infty(\mathcal{D})$. For $H = L^2(\mathcal{D})$ and $V = H_0^1(\mathcal{D})$, we introduce $A\colon V \to V^*$ and $B(u) \, \mathrm{d}W \in H$ as

$$\langle Au, v \rangle_{V^* \times V} = \int_{\mathcal{D}} \nabla u \cdot \nabla v \, \mathrm{d}x, \quad u, v \in V,$$

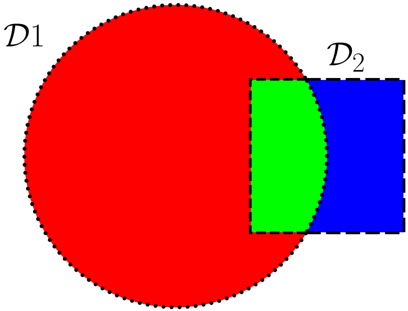
$$(B(u) \, \mathrm{d}W, v) = \int_{\mathcal{D}} B(u)v \, \mathrm{d}x \, \mathrm{d}W, \quad u, v \in H.$$

Then the SPDE can be written as an abstract equation

$$\begin{cases} \mathrm{d}u(t) + Au(t) \, \mathrm{d}t = B(u(t)) \, \mathrm{d}W(t) & \text{in } V^*, \quad t \in (0, T], \\ u(0) = u_0 & \text{in } H. \end{cases}$$

Decomposition of unity

- We consider the overlapping subdomains $\mathcal{D}_\ell$, $\ell = 1, \dots, s$, with $\bigcup_{\ell=1}^{s} \mathcal{D}_\ell = \mathcal{D}$.



- We introduce a partition of unity $(\chi_\ell)_{\ell=1,\dots,s}$ in $W^{1,\infty}(\mathcal{D})$.

- We assume that

$$\sum_{\ell=1}^{s} \chi_\ell = 1, \quad supp(\chi_\ell) \subseteq \mathcal{D}_\ell, \quad \chi_\ell(x) \in (0,1] \text{ for } x \in \mathcal{D}_\ell.$$

Suitable operators: Original problem

We find the weak formulation through

$$\int_{\mathcal{D}} \mathrm{d}uv \, \mathrm{d}x + \int_{\mathcal{D}} \nabla u \cdot \nabla v \, \mathrm{d}t \, \mathrm{d}x = \int_{\mathcal{D}} (B(u) \, \mathrm{d}W) v \, \mathrm{d}x$$

for $v \in C_0^\infty(\mathcal{D})$. For $H = L^2(\mathcal{D})$ and $V = H_0^1(\mathcal{D})$, we introduce $A \colon V \to V^*$ and $B(u) \, \mathrm{d}W \in H$ as

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$$\begin{cases} \mathrm{d}u(t) + Au(t) \, \mathrm{d}t = B(u(t)) \, \mathrm{d}W(t) & \text{in } V^*, \quad t \in (0, T], \\ u(0) = u_0 & \text{in } H. \end{cases}$$

Suitable operators: Decomposed problem

We find the weak formulation through

$$\int_{\mathcal{D}} \mathrm{d}uv \,\mathrm{d}x + \int_{\mathcal{D}_\ell} \chi_\ell \nabla u \cdot \nabla v \,\mathrm{d}x \,\mathrm{d}t = \int_{\mathcal{D}_\ell} \chi_\ell B(u) v \,\mathrm{d}x \,\mathrm{d}W.$$

for $v \in C_0^\infty(\mathcal{D})$. For $H = L^2(\mathcal{D})$ and $V_\ell \supset H_0^1(\mathcal{D})$, we introduce $A_\ell \colon V_\ell \to V_\ell^*$ and $B_\ell(u)\,\mathrm{d}W \in H$ as

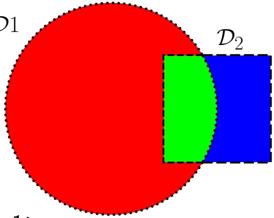
$$\langle A_\ell u, v\rangle_{V_\ell^* \times V_\ell} = \int_{\mathcal{D}_\ell} \chi_\ell \nabla u \cdot \nabla v \,\mathrm{d}x, \quad u, v \in V_\ell,$$

$$(B_\ell(u)\,\mathrm{d}W, v) = \int_{\mathcal{D}_\ell} \chi_\ell B(u) v \,\mathrm{d}x \,\mathrm{d}W, \quad u, v \in H.$$

Then the SPDE can be written as an abstract equation

$$\begin{cases} \mathrm{d}u(t) + A_\ell u(t)\,\mathrm{d}t = B_\ell(u(t))\,\mathrm{d}W(t) & \text{in } V_\ell^*, \quad t \in (0, T], \\ u(0) = u_0 & \text{in } H. \end{cases}$$

First scheme



Standard scheme:

$$\begin{cases} U^n - U^{n-1} + hA(t_n, U^n) = hF(t_{n-1}, U^{n-1}) \\ \qquad\qquad\qquad\qquad\qquad + B(t_{n-1}, U^{n-1})\Delta W^{n,K} & \text{in } V^* \\ U^0 = u_0 & \text{in } H. \end{cases}$$

Or equivalently

$$V^n = U^{n-1} + hF(t_{n-1}, U^{n-1}) + B(t_{n-1}, U^{n-1})\Delta W^{n,K}$$
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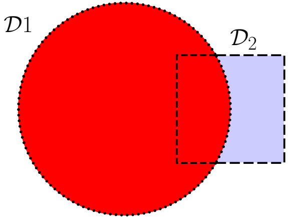
Domain decomposition scheme



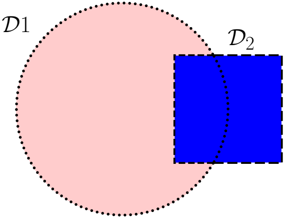
Now we consider:

$$\begin{cases} U_1^n - U^{n-1} + hA_1(t_n, U_1^n) = hF_1(t_{n-1}, U^{n-1}) \\ \qquad\qquad\qquad + B_1(t_{n-1}, U^{n-1})\Delta W^{n,K} & \text{in } V_1{}^*, \\ U^0 = u_0 & \text{in } H, \end{cases}$$

or equivalently

$$V_1^n = U^{n-1} + hF_1(t_{n-1}, U^{n-1}) + B_1(t_{n-1}, U^{n-1})\Delta W^{n,K}$$
$$U_1^n = (I + hA_1(t_n, \cdot))^{-1} V_1^n$$

Domain decomposition scheme



Now we consider:

$$\begin{cases} U^n - U_1^n + hA_2(t_n, U_2^n) = hF_2(t_{n-1}, U_1^n) \\ \qquad\qquad\qquad\qquad + B_2(t_{n-1}, U^{n-1})\Delta W^{n,K} & \text{in } V_2{}^*, \\ U^0 = u_0 & \text{in } H, \end{cases}$$

or equivalently

$$V_2^n = U_1^n + hF_2(t_{n-1}, U_1^n) + B_2(t_{n-1}, U^{n-1})\Delta W^{n,K}$$
$$U_1^n = (I + hA_1(t_n, \cdot))^{-1} V_2^n$$

Final scheme

We combine this scheme with an operator splitting

$$\begin{cases} U_\ell^n - U_{\ell-1}^n + hA_\ell(t_n, U_\ell^n) = hF_\ell(t_{n-1}, U_{\ell-1}^n) & \\ \qquad\qquad\qquad\qquad\qquad\qquad + B_\ell(t_{n-1}, U^{n-1})\Delta W^{n,K_\ell}, & \text{in } V_\ell^* \\ U^n = U_s^n, U_0^n = U_s^{n-1} & \text{in } H \\ U_0^1 =, U_0^1 = u_0 & \text{in } H, \end{cases}$$

for $\ell \in \{1, \dots, s\}$ or equivalently

$$V_\ell^n = U_{\ell-1}^n + hF(t_{n-1}, U_{\ell-1}^n) + B_\ell(t_{n-1}, U^{n-1})\Delta W^{n,K}$$
$$U_\ell^n = (I + hA_\ell(t_n, \cdot))^{-1} V_\ell^n.$$

This means:

- Update solution on first part of the domain.
- Use this solution as an initial value when updating the solution on second domain. And so on.
- After last domain, we obtain the final solution for the time-step.

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Explicit error bounds

Assume that

- $u \in C^\nu([0,T]; L^2(V,\Omega))$, $\nu \in (0, 1/2)$,
- regularity assumption on the decomposition,
- eigenvalues of Q fulfill: $\sum_{k>K} q_k < \varepsilon$.

Then the error $e^n = u(t_n) - U^n$ can be bound by

$$\mathbf{E}\left[\|e^n\|_H^2\right] + \sum_{i=1}^{n}\sum_{\ell=1}^{s}\mathbf{E}\left[\|e_\ell^i - e_{\ell-1}^i\|_H^2\right] + \sum_{i=1}^{n} h \sum_{\ell=1}^{s}\mathbf{E}\left[\|e_\ell^i\|_{V_\ell}^2\right]$$
$$\leq C\left(h^{2\nu} + \varepsilon\right).$$

Note:
Same error rate as in [Mukam, Tambue, 2019], [Kruse, Weiske, 2021].

Regularity of the solution

The mild solution of the semi-linear problem fulfills

$$\sup_{t_1,t_2\in[0,T],t_1\neq t_2} \frac{\left(\mathbf{E}\left[\|u(t_1)-u(t_2)\|^2_{H^1_0}\right]\right)^{\frac{1}{2}}}{\left|t_1-t_2\right|^{\min(\frac{1}{2},\frac{r}{2})}} < \infty$$

if additionally

- coefficients are time-independent ,
- $-A$ is the generator of an analytic semigroup of contractions (in particular linear),
- initial value lies in $L^2(\Omega, H^2(\mathcal{D}) \cap H^1_0(\mathcal{D}))$,
- Sufficient regularity in diffusion term: $\|A^{\frac{r}{2}}B(x)\|_{\mathcal{HS}(H)} \leq C(1+\|x\|_{H^r_0})$

Compare [Kruse, Larsson, 2012].

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Numerical example 1

For the domain $\mathcal{D} = (0,1)^2 \subset \mathbf{R}^2$, we consider the SPDE

$$\begin{cases} \mathrm{d}u(t,x) - D(t)\Delta u(t,x)\,\mathrm{d}t = F(t,x,u)\,\mathrm{d}t + B(t,x,u)\,\mathrm{d}W(t,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in (0,1], x \in \mathcal{D}, \\ u(t,x) = 0, \qquad\qquad\qquad\qquad\quad t \in [0,1], x \in \partial\mathcal{D}, \\ u(0,x) = u_0(x), \qquad\qquad\qquad\quad\ x \in \mathcal{D}. \end{cases}$$

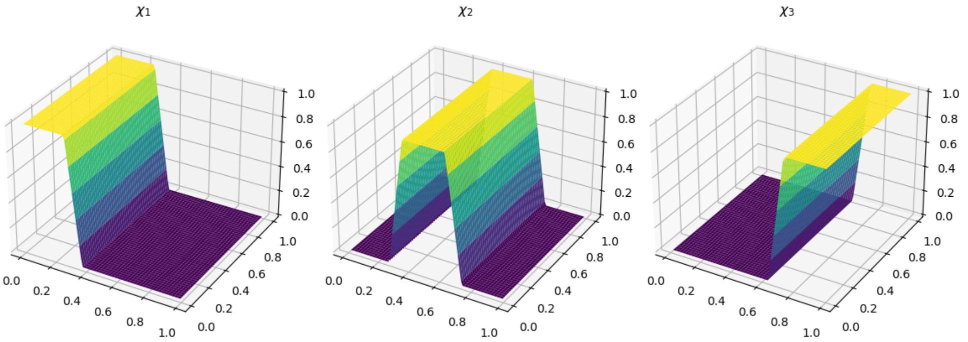
where

$$u_0(x) = 5x_1^2(x_1-1)^3 x_2^2(x_2-1)^3, \quad D(t) = 0.1(1+\exp(-t))$$
$$F(t,x,u) = \sin(u), \quad B(t,x,u) = 1.$$

This means, we have a semi-linear equation with additive noise.

Weight functions for domain decomposition

For the domain decomposition, we divide the domain into three part. For this we use the weight functions:



Parameters for experiment

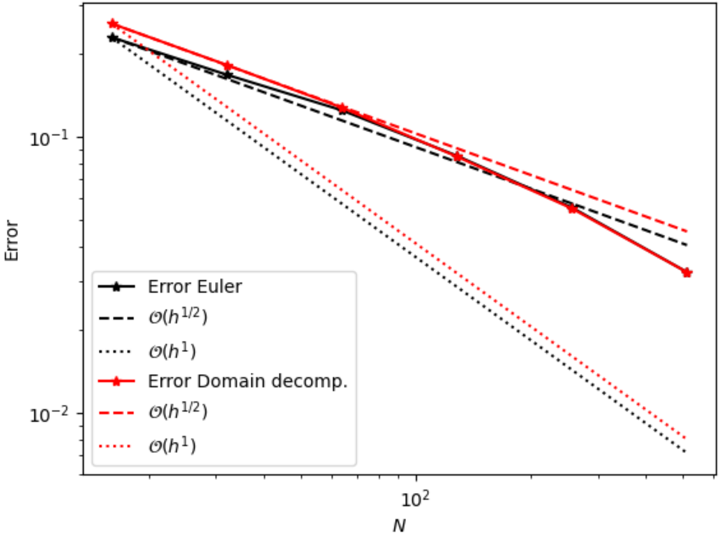
For the test we use the following parameters:

- Use weight functions χ_1, χ_2 and χ_3 from previous slide.
- Monte Carlo iterations: 10.
- For a family of Wiener processes $(\beta_{i,j})_{i,j}$, we use the truncated noise

$$W^K(t,x) = \sum_{i,j=1}^{K} (i+j)^{-\frac{1}{2}(2r+1+\varepsilon)} \beta_{i,j}(t) e_{i,j}(x)$$

  with regularity and truncation parameters $r = 0.1$ and $K = 100$.
- Spatial discretization parameter $h_{\text{space}} = 1/100$.

Error plot



10^{-1}
10^{-2}
Error
10^2
N
Error Euler
$\mathcal{O}(h^{1/2})$
$\mathcal{O}(h^1)$
Error Domain decomp.
$\mathcal{O}(h^{1/2})$
$\mathcal{O}(h^1)$

Numerical example 2

For the domain $\mathcal{D} = (0,1)^2 \subset \mathbf{R}^2$, we consider the SPDE

$$\begin{cases} \mathrm{d}u(t,x) - D(t)\Delta u(t,x)\,\mathrm{d}t = F(t,x,u)\,\mathrm{d}t + B(t,x,u)\,\mathrm{d}W(t,x), & \\ & t \in (0,1], x \in \mathcal{D}, \\ u(t,x) = 0, & t \in [0,1], x \in \partial\mathcal{D}, \\ u(0,x) = u_0(x), & x \in \mathcal{D}. \end{cases}$$

where

$$\begin{aligned} u_0(x) &= 5x_1^2(x_1-1)^3 x_2^2(x_2-1)^3, \quad D(t) = 0.1 \\ F(t,x,u) &= (3\sin(5x_1) + 2\cos(7x_2))(\cos(t) + \sin(4t)) \\ B(t,x,u) &= u. \end{aligned}$$

This means, we have a linear equation with multiplicative noise.

Weight functions for domain decomposition

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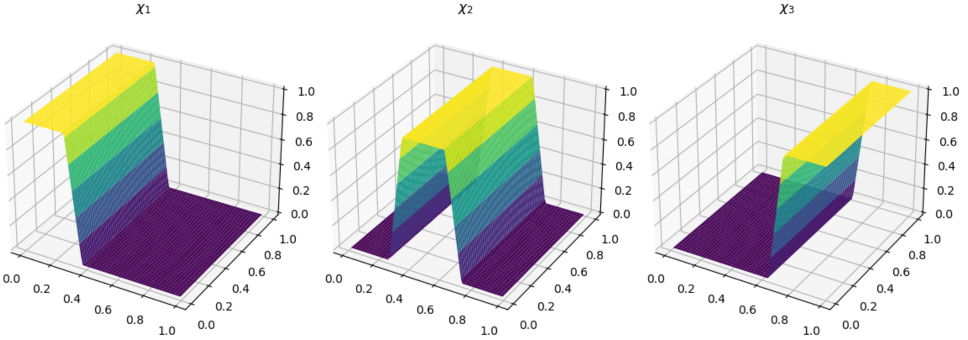

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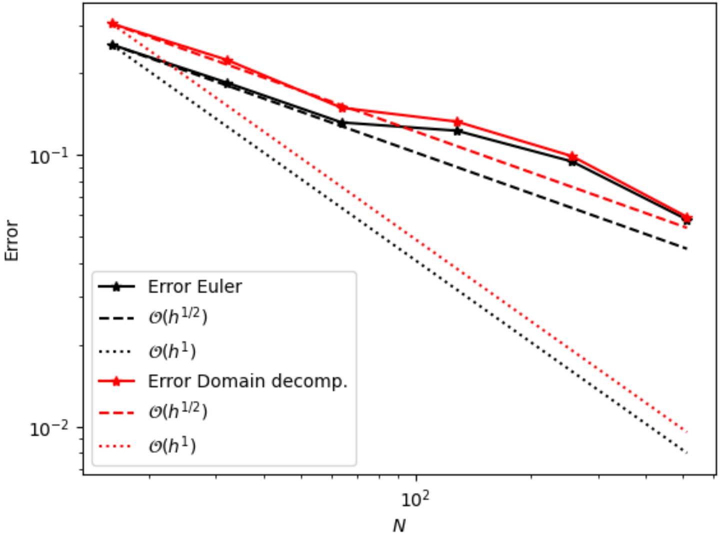

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$\mathcal{O}(h^{1/2})$
$\mathcal{O}(h^1)$

Thank you for your attention!