Properties of Marginal Sequential Monte Carlo Methods

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Marginal Particle Filters

Consider a state space model (SSM) $(X_n, Y_n)_{n\geq 0}$ with

- transition density for the *latent* process $f_n(x_n|x_{n-1})$
- likelihood for the *observation* process $g_n(y_n|x_n)$

We are interested in approximating the filtering distribution

$$p(x_n|y_{1:n}) \propto g_n(y_n|x_n) \int f_n(x_n|x_{n-1})p(x_{n-1}|y_{1:n-1})\mathrm{d}x_{n-1}.$$

Algorithm 1: Marginal Particle Filter [2]

1: For $i = 1, \dots, N$, sample

$$X_n^i \sim \sum_{j=1}^N W_{n-1}^j q_n\left(\cdot|y_n, X_{n-1}^j\right)$$

2: For $i = 1, \dots, N$, compute the importance weights

$$W_n^i \propto g_n(y_n|X_n^i)\frac{\sum_{j=1}^N W_{n-1}^j f_n(X_n^i|X_{n-1}^j)}{\sum_{j=1}^N W_{n-1}^j q_n(X_n^i|y_n, X_{n-1}^j)}$$

We establish

- strong law of large numbers

$$\sum_{i=1}^N W_n^i\, \varphi(X_n^i) \xrightarrow{a.s.} \int \varphi(x_n)p(x_n|y_{1:n})\mathrm{d}x_n$$

- central limit theorem

$$\sqrt{N}\left(\sum_{i=1}^N W_n^i\, \varphi(X_n^i) - \int \varphi(x_n)p(x_n|y_{1:n})\mathrm{d}x_n\right) \xrightarrow{d} \mathcal{N}(0, \mathrm{V}_n^{\mathrm{MPF}}(\varphi))$$

In particular, when using the same proposal q_n,

$$\boxed{\mathrm{V}_n^{\mathrm{MPF}}(\varphi) \leq \mathrm{V}_n^{\mathrm{PF}}(\varphi)}$$

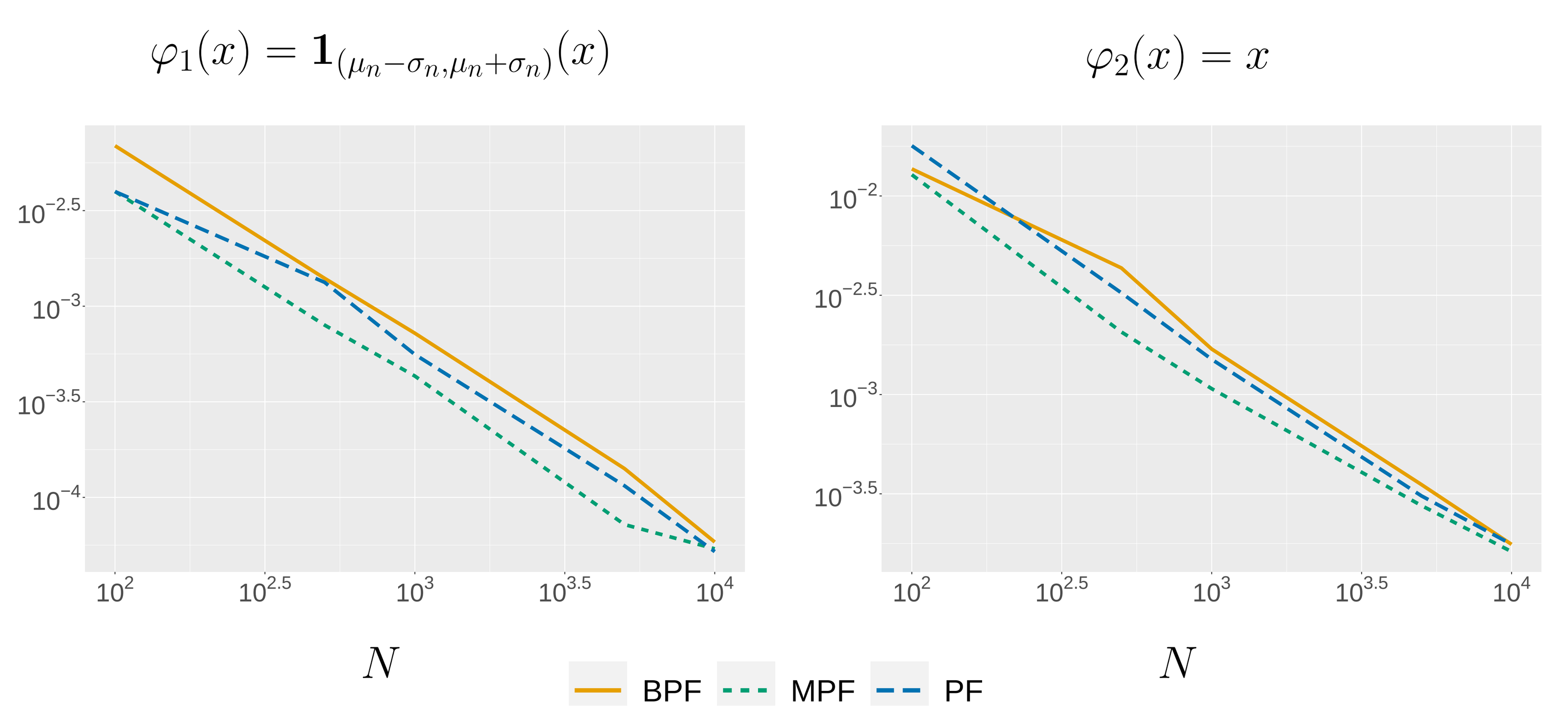

Fig. 1: Variance for BPF, PF and MPF using the locally optimal proposal.

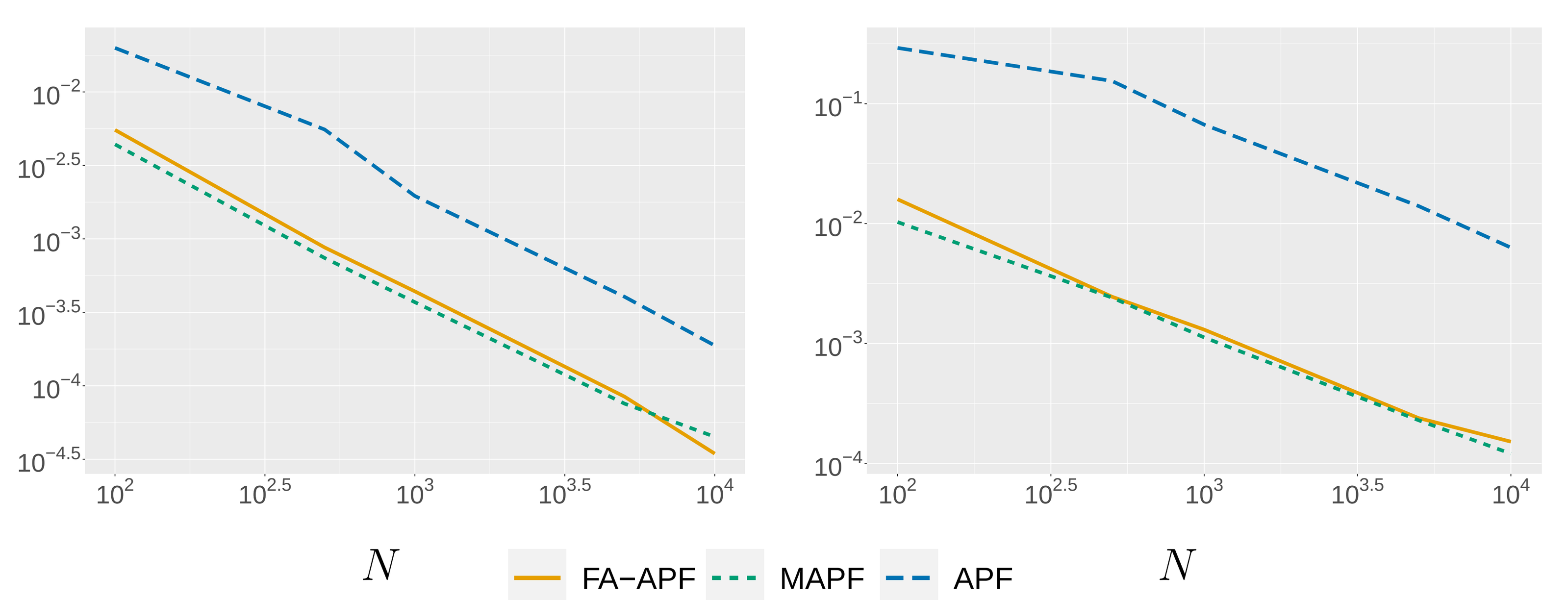

Fig. 2: Variance of FA-APF, APF and MAPF. The proposal has the same mean of the locally optimal proposal of FA-APF but twice the variance.

SMC for Approximate Bayesian Computation

ABC-SMC approximates the posterior distribution of a parameter θ when the likelihood function $p(y_{obs}|\theta)$ is intractable but can be sampled from.

- $p(\theta)$ a prior on the parameter θ
- $p(\cdot|\theta)$ the intractable likelihood
- π_{ϵ_n} a kernel with a degree of concentration determined by ϵ_n which measures how close y_n is to the observed data y_{obs}

We are interested in approximating the posterior distribution

$$p(\theta|y_{obs}) \propto p(\theta)p(y_{obs}|\theta)$$

using the approximate likelihood

$$\hat{p}_n(y_{obs}|\theta) = \int p(y_{obs}|\theta)\pi_{\epsilon_n}(y|y_{obs})\mathrm{d}y$$

Algorithm 2: ABC-SMC [3]

1: For $i = 1, \dots, N$, sample $\theta_n^i \sim q_n\left(\cdot|\theta_{n-1}^i\right)$ and simulate $Y_n^i \sim p\left(\cdot|\theta_n^i\right)$

2: For $i = 1, \dots, N$, compute the importance weights

$$\widetilde{W}_n^i = \frac{p(\theta_n^i)\pi_{\epsilon_n}(Y_n^i|y_{obs})}{\sum_{j=1}^N W_{n-1}^j q_n(\theta_n^i|\theta_{n-1}^j)}$$

3: Normalize the weights $W_n^i = \widetilde{W}_n^i / \sum_{j=1}^N \widetilde{W}_n^j$ and resample

We establish

- strong law of large numbers

$$\sum_{i=1}^N W_n^i\, \varphi(\theta_n^i) \xrightarrow{a.s.} \int \varphi(\theta_n)\hat{p}_n(\theta_n|y_{obs})\mathrm{d}\theta_n$$

- central limit theorem

$$\sqrt{N}\left(\sum_{i=1}^N W_n^i\, \varphi(\theta_n^i) - \int \varphi(\theta_n)\hat{p}_n(\theta_n|y_{obs})\mathrm{d}\theta_n\right) \xrightarrow{d} \mathcal{N}(0, \mathrm{V}_n^{\text{ABC-SMC}}(\varphi))$$

- unbiasedness

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \widetilde{W}_n^i\right] = \int\int p(\theta_n)p(y_{obs}|\theta_n)\pi_{\epsilon_n}(y_n|y_{obs})\mathrm{d}y_n\mathrm{d}\theta_n \approx p(y_{obs})$$

The last result shows that ABC-SMC provides estimates of the model evidence whose bias only depends on ϵ_n.

Conclusions

We provide a general framework to study marginal sequential Monte Carlo algorithms and we characterize their asymptotic behaviour [1]. This framework admits a number of algorithms as particular cases, including (a) marginal particle filters, marginal auxiliary particle filters, independent particle filters, (b) ABC-SMC, marginal SMC for doubly intractable models (c) score vector and observed information matrix estimation for particle filters.

References

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