Friedrich-Alexander-Universität Erlangen-Nürnberg

Friedrich-Alexander-Universität DYNAMICS, CONTROL, MACHINE LEARNING AND NUMERICS

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AN A POSTERIORI PROBABILISTIC ROBUSTNESS CHECK FOR DETERMINISTIC OPTIMAL CONTROLS

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Mathematics of Gas Flow in Pipeline Networks - The TRR 154

TRR 154 - Mathematical modelling, simulation and optimization using the example of gas networks

- The "turnaround in energy policy" is currently in the main focus of public opinion. It concerns social, political and scientific aspects as the dependence on a reliable, efficient and affordable energy supply becomes increasingly dominant. On the other side, the desire for a clean, environmentally consistent and climate-friendly energy production is stronger than ever.
- Gas Flow in Pipeline Networks covers various mathematical areas including modelling, simulation, optimization, network theory, control theory, uncertainty, mixed integer programming and much more. A deep understanding of various topics and an intense collaboration is necessary to deal with problems and applications related to mathematics of gas transport in pipeline networks.

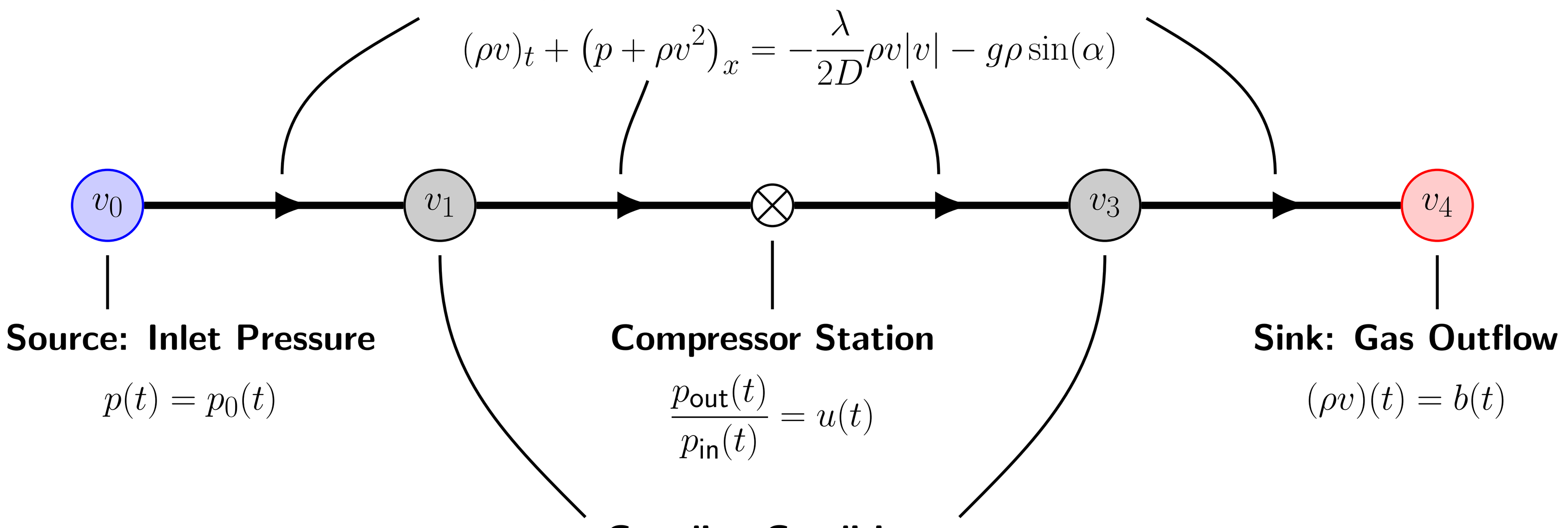
Mathematical Modelling of Gas Flow in Pipeline Networks

p	gas pressure	v	gas velocity	g	gravitational constant
ρ	gas density	λ/D	pipe friction	α	pipe slope

Isothermal Euler Equations

$$\rho_t + (\rho v)_x = 0$$

$$(\rho v)_t + \left(p + \rho v^2\right)_x = -\frac{\lambda}{2D}\rho v|v| - g\rho\sin(\alpha)$$



Source: Inlet Pressure

$p(t) = p_0(t)$

Compressor Station

$\frac{p_{\text{out}}(t)}{p_{\text{in}}(t)} = u(t)$

Sink: Gas Outflow

$(\rho v)(t) = b(t)$

Coupling Conditions

Conservation of Mass: $\sum(\rho v)_{\text{in}}(t) = \sum(\rho v)_{\text{out}}(t)$, Continuity in Pressure: $p_{\text{in}} = p_{\text{out}}$

Optimal Compressor Control with Buffer Zones on Gas Networks

Let bounds for the pressures $0 < p_{\min} < p_{\max}$ be given at every node. For $\varepsilon > 0$ consider the optimal control problem

$$\min_{u \in L^2(0,T)} f(u)$$

s.t.

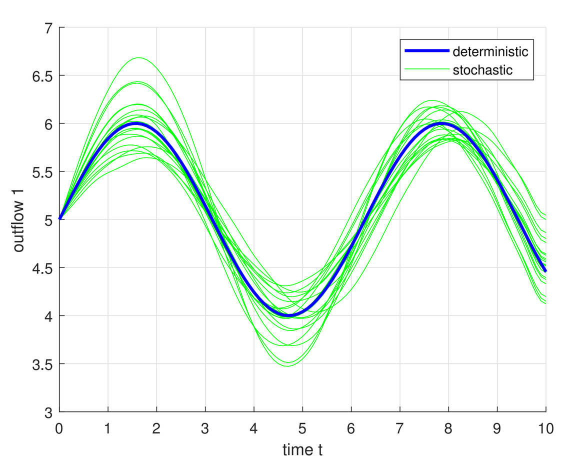
$\rho_t + (\rho v)_x = 0$, $(\rho v)_t + \left(p + \rho v^2\right)_x = -\frac{\lambda}{2D}\rho v\lvert v\rvert - g\rho\sin(\alpha)$	on every edge
$p(t) = p_0(t)$	on every source node
$(\rho v)(t) = b(t)$	on every sink node
$\sum(\rho v)_{\text{in}}(t) = \sum(\rho v)_{\text{out}}(t)$, $p_{\text{in}} = p_{\text{out}}$	on every inner node
$\frac{p_{\text{out}}(t)}{p_{\text{in}}(t)} = u(t)$	for every compressor
$p(t) \in \left[\, p_{\min} + \varepsilon, p_{\max} - \varepsilon \,\right] \quad \forall t \in [0,T]$	on every node

Uncertainty in the Gas Network

- Due to the structure of the European gas market, the consumers gas demand can be estimated but the exact demand is uncertain.
- We consider uncertain gas outflow $b^\omega(t)$ by randomizing the Fourier series of the estimated gas demand $b(t)$, i.e.,

$$b^\omega(t) = \sum_{m=0}^{\infty} \xi_m(\omega)\, a_m^0\, \psi_m(t),$$

$$a_m^0 = \int_0^T b(t)\,\psi_m(t), \quad \psi_m(t) = \sqrt{\frac{2}{T}}\sin\left(\left(\frac{\pi}{2} + m\pi\right)\frac{t}{T}\right)$$



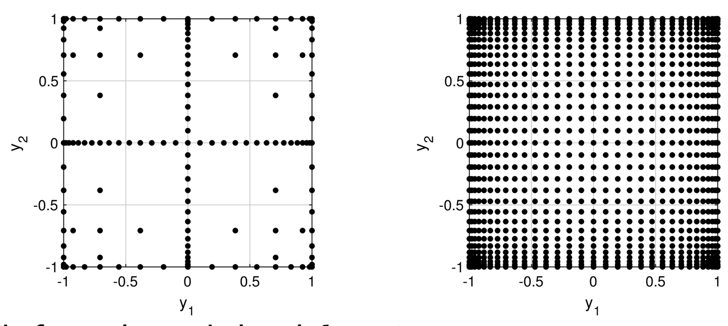
- Randomized Fourier series preserve regularity, so for $b \in L^2(0,T)$, that means we also have $b^\omega \in L^2(0,T)$ almost surely.
- Our aim is to compute the probability, that the pressure at the nodes corresponding to the random gas demand stays within the pressure bounds, i.e., for $\varepsilon > 0$ with corresponding optimal control $u(t)$ we compute the probability

$$\mathbb{P}\big(\, p(t) \in \left[\, p^{\min}, p^{\max} \,\right] \quad \forall t \in [0,T] \,\big).$$

Stochastic Collocation Method

- Let an optimal control $u(t)$, inlet pressures $p_0(t)$ and a n_S-dimensional random gas outflow $b^\omega(t)$ be given. We approximate the pressures $p(t,x,b^\omega)$ in the stochastic space by stochastic collocation on a *Smolyak sparse grid* with *Clenshaw-Curtis nodes.*
- For a multi-index $\mathbf{i} \in \mathbb{N}^{n_S}$ and a natural number $k \in \mathbb{N}$ we define the constant

$$c_{\mathbf{i}} = (-1)^{k+n_S-|\mathbf{i}|}\binom{n_S - 1}{k + n_S - |\mathbf{i}|}.$$



- The approximated pressures are given by the Smolyak formula with level $k > 0$

$$S_k[p(t,x,\,\cdot\,)] = \sum_{\substack{\mathbf{i} \in \mathbb{N}^{n_S} \\ k+1 \le \mathbf{i} \le k+n_S}} c_{\mathbf{i}}\left(\mathcal{U}^{(i_1)} \otimes \cdots \otimes \mathcal{U}^{(i_{n_S})}\right)[p(t,x,\,\cdot\,)],$$

where the $\mathcal{U}^{(i_m)}$ for $m = 1,\cdots,n_S$ are the interpolation operators on one dimension.

Kernel Density Estimation

- Let $\mathcal{P} = \{\, p_1(t), \cdots, p_{N_{\text{KDE}}}(t) \,\} \subseteq \mathcal{C}^0([0,T];\mathbb{R}^n)$ be an independent and identical distributed sample of (time dependent) pressures at the nodes.
- Let $\mathcal{P}_{\min}^{\max} = \{\, \underline{\overline{p}}_1, \cdots, \underline{\overline{p}}_{N_{\text{KDE}}} \} \subseteq \mathbb{R}^{2n}$ be the sample of minimal and maximal pressures of $\mathcal{P}$ over time, i.e.,

$$\underline{\overline{p}}_k = \begin{bmatrix} \min_{t\in[0,T]} p_k(t) \\ \max_{t\in[0,T]} p_k(t) \end{bmatrix}, \qquad k = 1,\cdots,N_{\text{KDE}}.$$

- Consider a multivariate Gaussian product kernel $K : \mathbb{R}^{2n} \to \mathbb{R}_+$ and a diagonal bandwidth matrix $H \in \mathbb{R}^{2n\times 2n}$ given by

$$K(z) = \frac{1}{(2\pi)^n}\prod_{j=1}^{2n}\exp\left(-\frac{1}{2}z_j^2\right) \quad \text{and} \quad H_{j,j} = h^2\,\Sigma_{j,j}, \quad h = \left(\frac{4}{(2n+2)N_{\text{KDE}}}\right)^{\frac{1}{2n+4}},$$

where Σ is the covariance matrix of the sample $\mathcal{P}_{\min}^{\max}$.

- Kernel density estimation provides an approximation for the probability density function of the pressure at the nodes given by

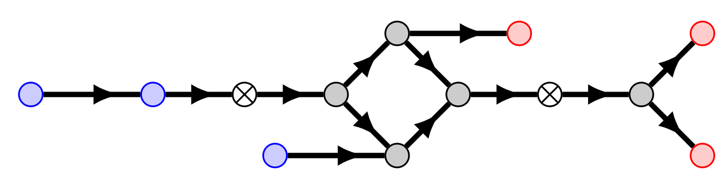
$$\rho_{p,N_{\text{KDE}}}(z) = \frac{1}{N_{\text{KDE}}\prod_{j=1}^{2n}\sqrt{H_{j,j}}}\sum_{i=1}^{N_{\text{KDE}}}\prod_{j=1}^{2n}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{z_j - \underline{\overline{p}}_{i,j}}{\sqrt{H_{j,j}}}\right)^2\right).$$

- Using the Gaussian error function erf, for the approximated probability we have

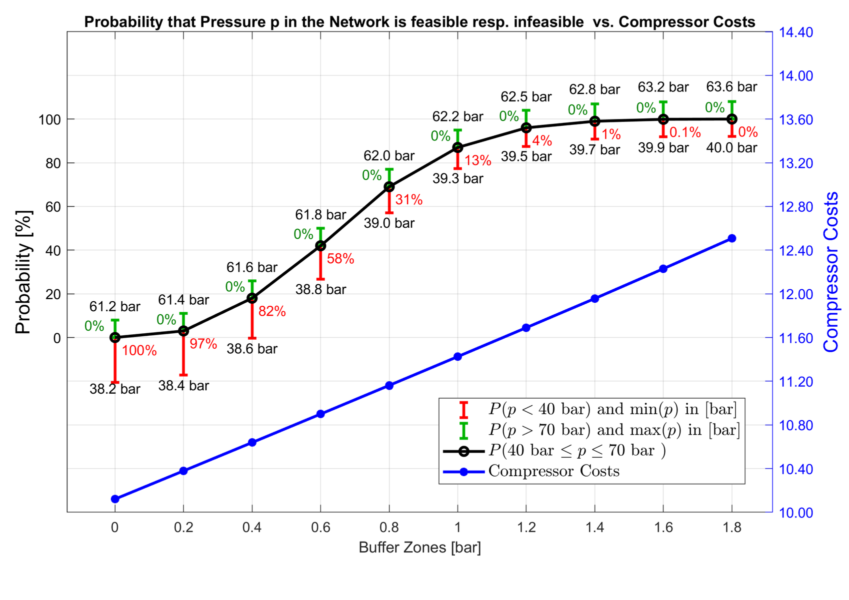
$$\mathbb{P}_{N_{\text{KDE}}}\big(\, p(t) \in \left[\, p^{\min}, p^{\max} \,\right] \forall t \in [0,T] \,\big) = \frac{1}{N_{\text{KDE}}\, 2^{2n}}\sum_{i=1}^{N_{\text{KDE}}}\prod_{j=1}^{2n}\left[\operatorname{erf}\left(\frac{p_j^{\max} - \underline{\overline{p}}_{i,j}}{\sqrt{2\,H_{j,j}}}\right) - \operatorname{erf}\left(\frac{p_j^{\min} - \underline{\overline{p}}_{i,j}}{\sqrt{2\,H_{j,j}}}\right)\right].$$

Probabilistic Robustness Check - Numerical Example

- Application to real network instance with data close to reality from *https://gaslib.zib.de/*:



- Stochastic collocation increases speed of computation enormously
- Kernel density estimation provides information on the probability density function and allows to compute derivatives of the probability



Selected publications

[1] Schuster, M., Strauch, E., Lang, J., Gugat, M. (2023) **An a posteriori Probabilistic Robustness Check for Deterministic Optimal Controls.** PREPRINT

[2] Schuster, M., Strauch, E., Gugat, M., Lang, J. (2022). **Probabilistic Constrained Optimization on Flow Networks.** Optim. Eng. 23, pp. 1–50.

[3] Schuster, M. (2021). **Nodal Control and Probabilistic Constrained Optimization using the Example of Gas Networks.** Dissertation, FAU Erlangen-Nürnberg, Germany 2021.

[4] Strauch, E. (2022). **Adaptive Multi-Level Monte Carlo and Stochastic Collocation Methods for Hyperbolic Partial Differential Equations with Random Data on Networks.** Dissertation, TU Darmstadt, Germany 2022.

https://dcn.nat.fau.eu/ 06/2023

TRR 154

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