Spatio-temporal modeling with SPDE based GMRF

The evolution of extreme rain phenomena in Austria

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1. Real-life problem

This study aims to describe the use of the **INLA-SPDE Bayesian approach** by using **Generalized Additive Models** to predict the changes in extreme weather phenomena as **maximum precipitation** and **dry spells** on a high-resolution map over the years 1972-1982 and 2012-2022 in Austria. Our spatio-temporal model is based on an **additive framework** to assess the effect of explanatory variables: elevation and latitude, resulting in a vector of covariates x_t and the **spatio-temporal random effect** z_t on the data y_t. The spatial dependency follows a Matérn model, and the temporal dependency an AR(1) process. Rewriting the model in a hierarchical form helps to represent all the spatio-temporal variability that may have an effect on the rain data y_t. The goal is to estimate parameters and interpret climatic variability over each 10 year period for different rain patterns.

2. Capturing dependencies

Let t represent time and s the spatial variable. Whittle noted in 1954 and 1963 that the **solution** $z(s,t)$ of the linear fractional stochastic partial differential equation (SPDE)

$$\frac{\partial}{\partial t}(\kappa(s)^2 - \Delta)\tau(s)z(s,t) = \mathcal{W}(s,t) \tag{1}$$

with $(s,t) \in \mathbb{R}^2 \times \mathbb{R}$, and spatially varying $\kappa(s)$ and $\tau(s)$ is a **non-stationary Gaussian random field (GRF) with Matérn covariance function**. The Matérn covariance of two points which are a distance r apart is given by:

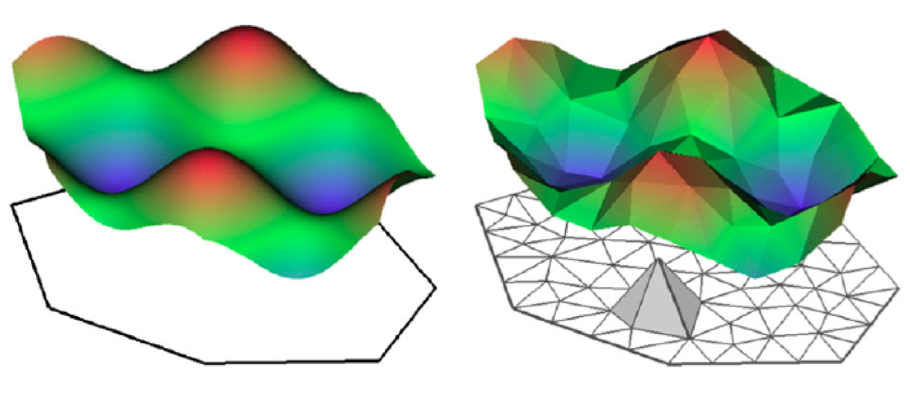
$$\Sigma := C(r) = \frac{\sigma_C^2}{\Gamma(1)}(\kappa||r||)K_1(\kappa||r||).$$

In 2011 Lindgren et.al. found an explicit link between the GRF and a **Gaussian Markov random field (GMRF)**. The key point is that the covariance function and the dense covariance matrix of a GRF are substituted, respectively, by a neighbourhood structure, see the picture below, and by a sparse precision matrix, $Q_S = \Sigma^{-1}$, that together define a GMRF. The GRF is then approximated by:

$$z(s,t) \approx \tilde{z}(s,t) = \sum_{j=1}^{N} \psi_j(s,t) z_j, \tag{2}$$

with basis functions $\psi_j(s,t) = \psi_j^s(s)\psi_j^t(t)$ and $z = (z_1, \dots, z_N)^T$ a collection of Gaussian weights with $z \sim N(0, Q^{-1})$ and a separable covariance matrix $Q = Q_T \otimes Q_S$.

The solution $z(s,t)$ is what allows to efficiently compute the spatial and temporal autocorrelation structure of a dataset by using the **separablility ansatz for covariance functions**: in the data models we account for spatial dependencies through the precision matrix Q_S and for temporal dependencies through an **AR(1) process**, represented by its precision matrix Q_T.



left: continuously indexed spatial GRF
right: corresponding FE representation

3. Generalized additive models for the data

Assume the rain data $y(s_i,t)$ is measured at location s_i, $i = 1, \dots, n$ and time $t = 1, \dots, T$ and, as all other variables, stored in a vector, i.e. $y_t := (y(s_1,t), \dots, y(s_n,t))^T$. The **generalized additive model** for the data is given by a link function g relating the expected value μ_t of y_t to the linear predictor η_t via

$$g(\mu_t) = \eta_t = \alpha + x_t\beta + z_t, \tag{3}$$

$$z_t = az_{t-1} + w_t, \quad w_t \sim N(0, Q_S^{-1}), \quad |a| < 1 \tag{4}$$

where x_t denotes the vector of covariates, w_t spatially correlated innovations and z_t represents (2) with $z_t \sim N(0, Q^{-1})$ and $Q = Q_T \otimes Q_S$. The observations y_t are modelled differently for each setup:

1. **Maximum precipitation** with blended generalized extreme value distribution:

$$y_t \sim \text{bGEV}(\eta_t, \log(\sigma), \text{tail}).$$

2. **Dry spells** with binomial distribution:

$$y_t \sim \text{Bin}(\text{Ntrials}, \text{logit}^{-1}(\eta_t)).$$

4. Integrated Nested Laplace Approximations (INLA)

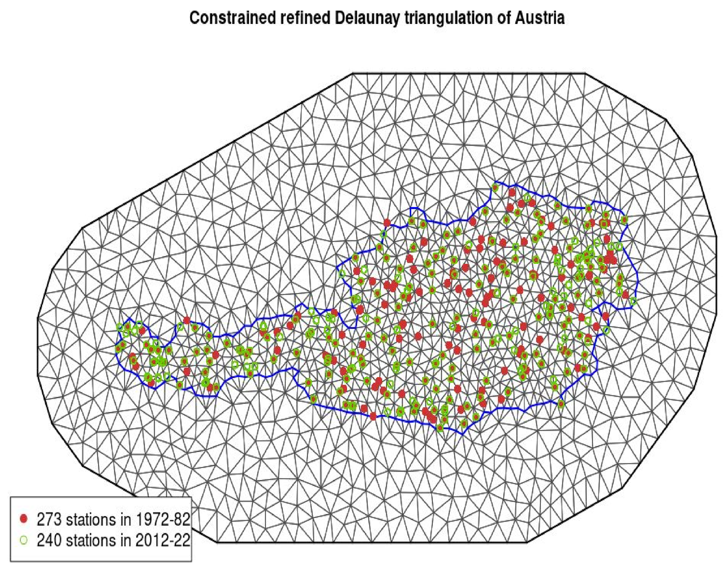
For **Bayesian inference** we rewrite the models above in a hierarchical form which can then be estimated using the **INLA** algorithm proposed by Rue et al. in 2009. INLA is an effective alternative to MCMC methods for **latent Gaussian field models**. Consider that the vector of latent effects $\xi_t = \{\eta_t, \alpha, \beta, z_t\}$ has the structure of a GMRF and $\theta_{\text{Bin}} = \{\sigma_w^2, \sigma_e^2, a, \kappa, \tau\}$ or $\theta_{\text{bGEV}} = \{\sigma_w^2, \sigma_e^2, a, \kappa, \tau, \sigma, \text{tail}\}$ represent the hyperparameter vectors for the respective setup. The joint posterior distribution is given by

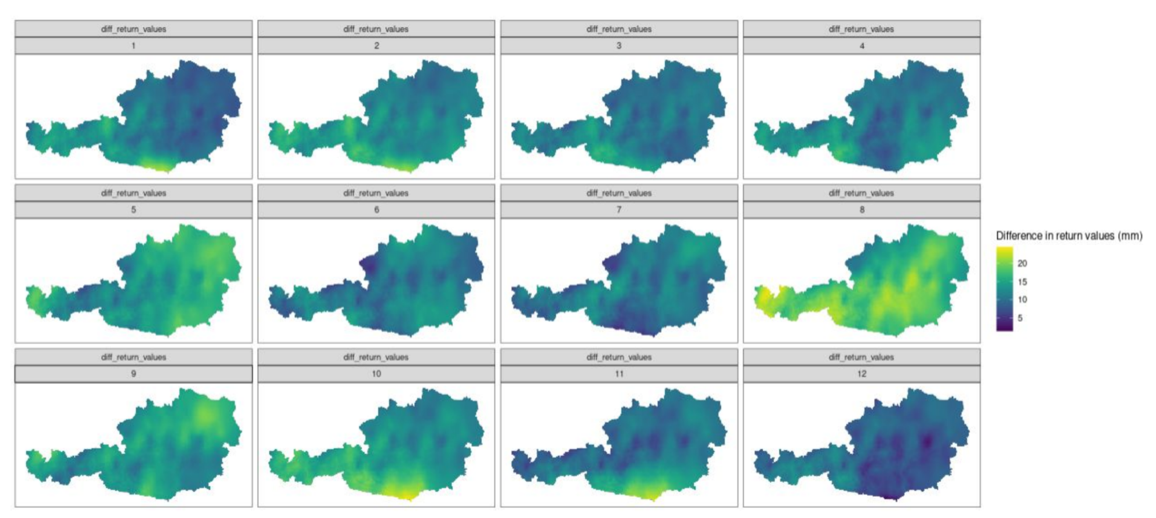
$$\pi(\xi_t, \theta | y_t) \propto \pi(\theta)\pi(\xi_t|\theta) \prod_{t=1}^{T} \pi(y_t|\xi_t, \theta).$$

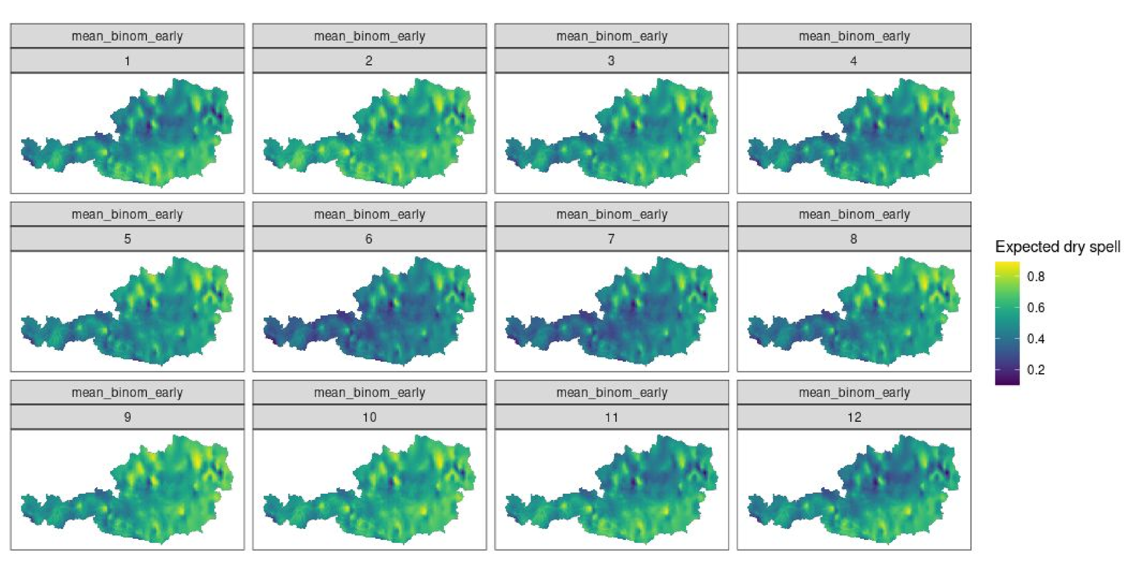
However, we are interested in the **marginal distributions** of the latent field and of the hyperparameters, i.e $\pi(\xi_t|y_t)$ and $\pi(\theta_i|y_t)$.

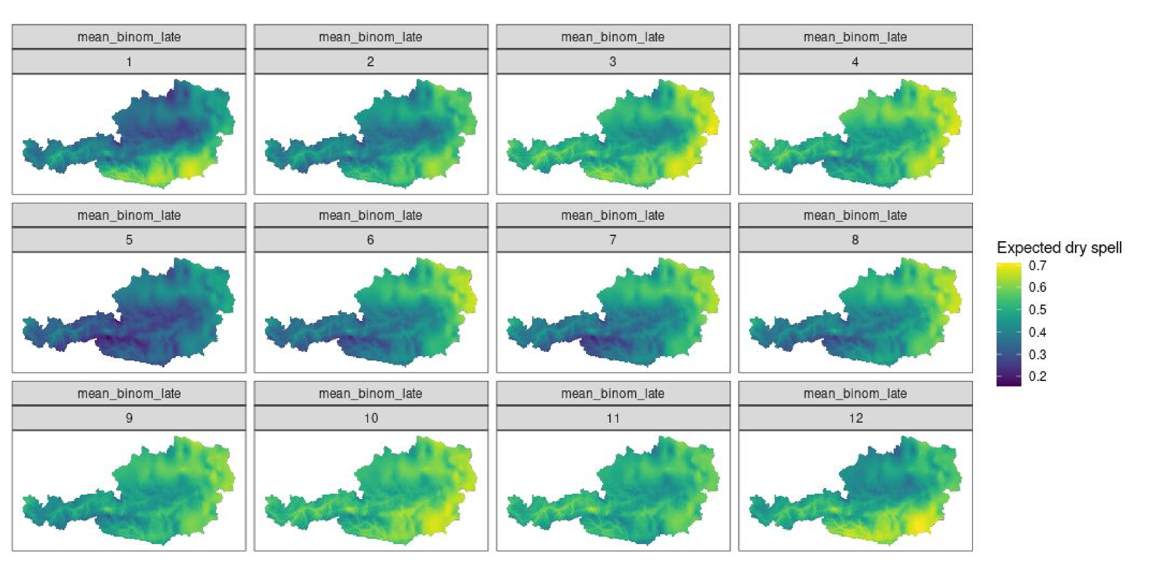
5. Results

First **discretise space** through a mesh that would create an artificial set of neighbours so we can calculate the autocorrelation between points through (1). After triangulating Austria with 1385 vertices, the main interest resides in **spatial prediction** of η_t at **unobserved locations**, i.e. below the posterior predictive mean of dry spells for the early (1972-1982) and late (2012-2022) time period. For maximum precipitation we computed the difference in **return values** of a 50 year return period.









6. References

Lindgren F., et. al., J R Stat Soc Series B Stat Methodol, Vol 73, 423–498, 2011.
Rue H., et. al., J R Stat Soc Series B Stat Methodol, Vol 71, 319–392, 2009.